NAVAL ARCHITECTURE.

BY

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PREFACE.

THE intention of this work is to give in a consistent and connected form the commonly accepted theory of naval architecture. So far as possible the treatment is simple and direct, especially for such subjects as displacement, stability, propulsion, and strength. A satisfactory treatment of some other parts, like waves and rolling of ships, is necessarily somewhat more intricate, but these chapters may be passed over in the first reading, or may be omitted altogether.

The book opens with a statement of rules for computing areas and a discussion of graphical integration and of mechanical integrators. After this introduction the subjects of displacement and stability are given in a simple manner, including the fundamental principles and methods of computation together with forms and examples. Afterward there follows a complete discussion of the surfaces of buoyancy and of water-lines, to give a firm grasp and broad view of the subjects treated in the preceding chapters; but this discussion is made to stand distinctly by itself so that it can be passed over without invalidating the work that precedes or follows it. The methods for determining displacement and stability are extended and applied to such general problems as adding and moving weights, grounding, docking, and launching.

A simple form of modern hydrodynamics is developed and applied to the theory of waves, including the common or trochoidal theory, the theory of irrotational waves, together with the effect of surface tension, and a statement of Scott-Russell's solitary wave. A discussion is given of the rolling of ships in an unresisting medium, in quiet water, and among waves, including experiments on resistance

to rolling and the application of the results of such experiments to the determination of the probable maximum rolling of ships at sea.

The treatment of the subjects of resistance and propulsion of ships is founded, as indeed it must be, on the experiments of the Froudes. Advantage is taken of the methods for designing screw propellers that have been developed by Naval Constructor Taylor and by Mr. Sidney Barnaby; extensive tables for the application of Taylor's method have been computed for this work. A discussion is given of steering and sailing.

Methods are given for the computation of stresses in ships both in quiet water and among waves; also for the computations for bulkheads. Attention is directed to those features which are peculiar to the computations for ships, and it is assumed that the reader is familiar with applied mechanics and the strength of materials.

During the development and preparation of this work a careful study was made of standard works on naval architecture, especially by French and English writers, and also of original articles and memoirs in scientific periodicals and in the transactions of scientific institutes and societies. Copious references will be found in the text to the original authorities quoted. While this work is intended primarily for students, it is hoped that it may be found useful by naval architects and shipbuilders in general.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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CHAPTER I.

INTEGRATION AND INTEGRATORS.

THEORETICAL naval architecture deals with the design and nautical properties of ships, including the discussion of displacement, stability, steadiness, strength, propulsion, and manœuvring, the methods of construction are usually described under the head of shipbuilding.

The form of a ship, though symmetrical and fair, is more or less arbitrary, and the lines, or horizontal and vertical sections, are bounded by smooth curves that cannot be represented by simple analytic equations. Consequently the integrations required for the determinations of areas, volumes, and other properties are made either by certain approximate methods or else by the aid of mechanical integrators. The approximate methods involve the use of certain rules, like the trapezoidal rule or Simpson's rule. An integrator is an instrument which measures or determines the area, moment, or moment of inertia of a figure of which the contour is traced by a tracing-point.

Trapezoidal Rule.—Let Fig. 1 represent a figure bounded by a continuous curve FHK, a base line AE, and two rectangular ordinates FA and KE at the ends. Let the base line be divided into any convenient number of equal parts each of which has the length s; let the several ordinates a, b, c, d, and e be measured. Then the figure bounded by the base line, the end ordinates, and a broken line

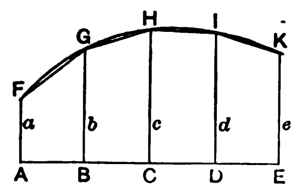

FIG. 1.

FGHIK will have the area

$$A = s\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+d}{2}+\frac{d+e}{2}\right)$$

$$= s(\tfrac{1}{2}a+b+c+d+\tfrac{1}{2}e). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The area bounded by the smooth curve is approximately represented by the same expression, the approximation being closer for a flat curve and for a large number of intervals. If the curve has points of inflection, so that it lies partly within and partly outside of the broken line which joins the ends of the ordinates, the error in excess for the first part may be compensated, partially or wholly, by the deficit in the other part.

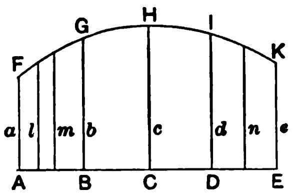

FIG. 2.

If the curve is sharper at the ends, intermediate ordinates may be drawn as in Fig. 2, dividing the end spaces into halves or into quarters. With half-intervals at the ends the area is given by the rule

$$A = s\left(\frac{a+m}{4}+\frac{m+b}{4}+\frac{b+c}{2}+\frac{c+d}{2}+\frac{d+n}{4}+\frac{n+e}{4}\right)$$

$$= s(\tfrac{1}{4}a+\tfrac{1}{2}m+\tfrac{3}{4}b+c+\tfrac{3}{4}d+\tfrac{1}{2}n+\tfrac{1}{4}e).$$

With quarter-intervals at one end the area is

$$A = s\left(\frac{a+l}{8}+\frac{l+m}{8}+\frac{m+b}{4}+\frac{b+c}{2}+\frac{c+d}{2}+\frac{d+n}{4}+\frac{n+e}{4}\right)$$

$$= s(\tfrac{1}{8}a+\tfrac{1}{4}l+\tfrac{3}{8}m+\tfrac{3}{4}b+c+\tfrac{3}{4}d+\tfrac{1}{2}n+\tfrac{1}{4}e). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is apparent that the trapezoidal rule may be used with any number of intervals, and that any interval at an end or elsewhere may be subdivided at will; and further. that a half- or quarter-interval may be added at an end or interpolated in the middle, provided the rule is modified to correspond. Though the rule is written for intervals of equal length (with subdivisions if desired). the same method may be applied with unequal intervals with somewhat more trouble.

Simpson's Rule.—Let the base be subdivided into an even number of parts so that there are an odd number of ordinates. Draw a line LN connecting the ends of the first and third ordinates, and a line HK parallel to LN through the end I of the second ordinate, intersecting the first and third ordinates at H and K. If the curve LIN were the arc of a parabola tangent at I to HK, then the area of the segment $LINM$ would be two thirds of the area of the parallelogram $HKNL$; portions of ships' lines between stations, or ordinates, nearly fulfil the conditions assumed.

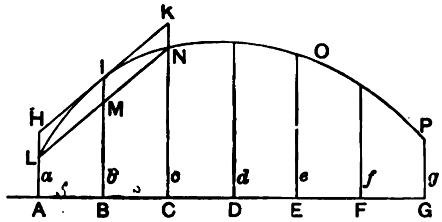

FIG. 3.

The area of the figure $ALINC$ between the curve, the base line, and the first and third ordinates is very nearly equal to

$$2s\frac{a+c}{2}+\frac{2}{3}2s\left(b-\frac{a+c}{2}\right)=\frac{1}{3}s(a+4b+c).$$

In like manner the areas between the third and fifth ordinates and the fifth and seventh ordinates are very nearly equal to

$$\tfrac{1}{3}s(c+4d+e) \quad \text{and} \quad \tfrac{1}{3}s(e+4f+g),$$

and the entire area between the curve LNP, the base line, and the end ordinate is

$$A=\tfrac{1}{3}s(a+4b+2c+4d+2e+4f+g) \qquad . \quad (3)$$

It is apparent that Simpson's rule may be used with any odd number of ordinates.

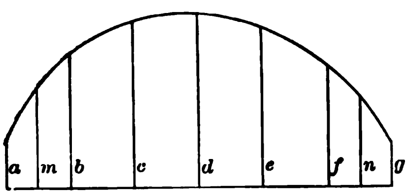

FIG. 4.

If the curve is sharper at the ends, half-intervals may be used. The area between the ordinates b and f may be calculated by the expression

$$\tfrac{1}{3}s(b+4c+2d+4e+f),$$

and the area between the ordinates a and b may be calculated, with

aid of an intermediate ordinate m, by the expression

$$\frac{1}{3}\cdot\frac{1}{2}s(a+4m+b),$$

while the area between the ordinates f and g may be calculated by the expression

$$\frac{1}{3}\cdot\frac{1}{2}s(f+4n+g).$$

The entire area with half-intervals at the ends may be found by the equation

$$A=\frac{1}{3}s(\tfrac{1}{2}a+2m+\tfrac{3}{2}b+4c+2d+4e+\tfrac{3}{2}f+2n+\tfrac{1}{2}g). \quad . \quad . \quad (4)$$

It is apparent that the number of whole intervals must be even, but that any number of pairs of half-intervals may be used at either end. Again, by proper modification the rule may be used with pairs of half-intervals at the middle.

Quarter-intervals may be provided by an extension of the method used for half-intervals. If the space between a and b, Fig. 4, were divided into fourths with the ordinates a, s, m, t, and b, the area of the figure between a and b would be

$$\frac{1}{3}\cdot\frac{1}{4}s(a+4s+2m+4t+b)=\frac{1}{3}s\left(\frac{a}{4}+s+\frac{m}{2}+t+\frac{b}{4}\right),$$

and the expression for the entire area would be

$$A=\frac{1}{3}s\left(\frac{a}{4}+s+\frac{m}{2}+t+\frac{5}{4}b+4c+2d+4e+\frac{3}{2}f+2n+\frac{1}{2}g\right). \quad . \quad (5)$$

Five-Eight Rule.—A special application of the method for Simpson's rule may be made to determine the area between any two successive ordinates provided a third ordinate is given. Let a, b, and c be ordinates with the interval s; then the area of the figure $ABED$ is made up of the area of the trapezoid $ADGB$ and the area of the half-segment DEG; the area of the trapezoid is

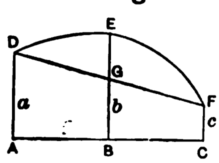

FIG. 5.

$$\frac{1}{2}s(AD+BG)=\frac{1}{2}s\left(AD+\frac{AD+FC}{2}\right)=\frac{s}{4}(3a+c);$$

the area of the half-segment is

$$\frac{1}{2}\times\frac{2}{3}\times 2s\left(b-\frac{a+c}{2}\right)=\frac{s}{3}(2b-a-c);$$

the combined areas of the trapezoid and the half-segment give

$$A=\frac{s}{12}(5a+8b-c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation is known as the five-eight rule.

Durand's Rules.—It has been found by Professor Durand * that the following equations have substantially the accuracy of Simpson's rule and are but little more troublesome to use than the trapezoidal rule:

$$\text{Area}=s(\tfrac{5}{12}a+\tfrac{13}{12}b+c+d+e+\tfrac{13}{12}f+\tfrac{5}{12}g).$$
$$\text{Area}=s(0.4a+1.1b+c+d+e+1.1f+0.4g).$$

The first rule is said to be somewhat more accurate, the second is clearly the easier to use.

Tchebycheff's Rule.—In the development of the preceding rules the ordinates are spaced at equal intervals; some of the ordinates are affected by special multipliers, and then these products and all

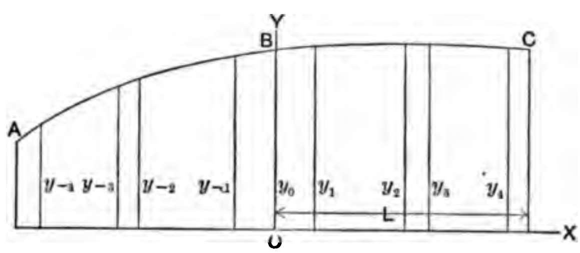

FIG. 6.

the other ordinates are added and the sum is multiplied by the distance between ordinates to get the area. In the trapezoidal rule the end ordinates are affected by multipliers; in Durand's rules several ordinates near the ends are affected; while for Simpson's rule all

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* *Engineering News* for Jan. 18, 1894. Trans. Soc. Nav. Arch. and Marine Eng., 1895, vol. 3, p. 127.

the ordinates are affected by multipliers. The subdivision of spaces at the ends for any of the rules modifies only the number of ordinates affected and their multipliers. It was suggested by Gauss that a determinate unequal spacing could be given to the ordinates so that they might be summed up directly. This has been done by Tchebycheff, giving the rule that is known by his name. To apply this rule to the determination of the area between a curve and an axis, as in Fig. 6, the origin is taken at the middle of the base and the ordinates are spaced symmetrically on the two sides of the origin; it should be noted that the ordinates at the ends of the curve A and C do not enter into the computation. After the ordinates are measured their sum is to be multiplied by the length of the base, and the product is to be divided by the number of ordinates to get the area. If there is an ordinate measured at the origin, there will be an odd number of ordinates; otherwise the number will be even. The following table gives the spacing of the ordinates from the middle of the base.

SPACING OF ORDINATES FOR TCHEBYCHEFF'S RULE.

Number of Ordinates.	Positions of Ordinates from Middle of Base in Fractions of Half-length of Base.
2	0.5773
3	0 0.7071
4	0.1876 0.7947
5	0 0.3745 0.8325
6	0.2666 0.4225 0.8662
7	0 0.3239 0.5297 0.8839
9	0 0.1679 0.5288 0.6010 0.9116

To determine the spacing of ordinates, let it be assumed that the curve ABC, Fig. 6, is replaced by a curve represented by the equation

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8, \quad . \quad (1)$$

where a_0, a_1, a_2, etc., are arbitrary constants.

The area between the curve represented by equation (1) and the axis of x may be obtained by integration as follows, taking L to represent half the length of the base:

$$\text{Area} = \int_{-L}^{L} y dx = 2L\left(a_0 + a_2\frac{L^2}{3} + a_4\frac{L^4}{5} + a_6\frac{L^6}{7} + a_8\frac{L^8}{9}\right). \quad . \quad (2)$$

But Tchebycheff's rule requires that the area shall be represented by the equation

$$\text{Area} = \frac{2L}{9}(y_0 + y_1 + y_{-1} + y_2 + y_{-2} + y_3 + y_{-3} + y_4 + y_{-4}). \quad . \quad (3)$$

Expressions for the several ordinates in equation (3) can be obtained from equation (1) as follows:

$$y_0 = a_0,$$
$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_1^4 + a_5x_1^5 + a_6x_1^6 + a_7x_1^7 + a_8x_1^8,$$
$$y_{-1} = a_0 - a_1x_1 + a_2x_1^2 - a_3x_1^3 + a_4x_1^4 - a_5x_1^5 + a_6x_1^6 - a_7x_1^7 + a_8x_1^8,$$

and six other equations in terms of y_2, y_{-2}, y_3, y_{-3}, y_4, and y_{-4}.

If the ordinates in equation (3) are replaced by their values, the resultant equation after reduction is

$$\text{Area} = \frac{2L}{9}\left\{ 9a_0 + 2a_2(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 2a_4(x_1^4 + x_2^4 + x_3^4 + x_4^4) \right.$$
$$\left. + 2a_6(x_1^6 + x_2^6 + x_3^6 + x_4^6) + 2a_8(x_1^8 + x_2^8 + x_3^8 + x_4^8) \right\}. \quad (4)$$

Equating the coefficients of a_2, a_4, a_6, and a_8 in equations (2) and (4), we have

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= \tfrac{9}{6}L^2, \\ x_1^4 + x_2^4 + x_3^4 + x_4^4 &= \tfrac{9}{10}L^4, \\ x_1^6 + x_2^6 + x_3^6 + x_4^6 &= \tfrac{9}{14}L^6, \\ x_1^8 + x_2^8 + x_3^8 + x_4^8 &= \tfrac{9}{18}L^8. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These four simultaneous equations lead to four equations of the fourth degree in L^2, from which the values of the abscissæ x_1, x_2, x_3, and x_4 can be determined. The general solution has been carried through thus far for nine ordinates (one being at the origin), which is the greatest number appearing in Tchebycheff's table. If there are less than nine ordinates, the corresponding set of equations can be obtained from (5) by omitting redundant terms and replacing 9 in the coefficients of terms containing L by the proper number of ordinates. Thus for five ordinates we have

$$x_1^2 + x_2^2 = \tfrac{5}{6}L^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$
$$x_1^4 + x_2^4 = \tfrac{5}{10}L^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Squaring equation (6) and subtracting (7) gives

$$2x_1^2x_2^2=\tfrac{7}{36}L^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Solving for x_2^2 and subtracting from equation (6) gives

$$x_1^2=\frac{5}{6}L^2-\frac{7}{72}\frac{L^4}{x_1^2},$$

whence

$$\frac{x_1}{L}=\left\{\frac{5}{12}-\left(\frac{25}{144}-\frac{7}{72}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}=0.3745,$$

and in a similar way

$$\frac{x_2}{L}=\left\{\frac{5}{12}+\left(\frac{25}{144}-\frac{7}{72}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}=0.8325.$$

Comparison of Rules.—With the same number of ordinates or stations, it is clear that the trapezoidal rule is more expeditious than Simpson's rule, but for the same degree of accuracy the trapezoidal rule should have half again as many stations and water-lines. Since both water-lines and transverse sections are treated by the rule chosen, in order to compute the displacement, the number of operations for the trapezoidal rule will be about $(\frac{3}{2})^2$ times, or about twice as numerous as those for Simpson's rule. Again, Simpson's rule gains materially in accuracy by using half-stations at the ends of water-lines; eight intervals with half-intervals at the ends are nearly as good as sixteen intervals.

If a special set of lines must be drawn for computation, the fewer number of water-lines and stations gives a considerable advantage to Simpson's rule. But there are commonly enough stations and water-lines drawn in fairing the ship for either rule, and the trapezoid rule has the advantage that unequal intervals may be used with a little extra trouble.

Tchebycheff's rule with nine ordinates is said to give as close an approximation as Simpson's rule with seventeen ordinates; as the stations must be spaced unevenly to correspond with the rule, it is

apparent that a special set of ordinates must be located and measured in determining the displacement of a ship. This rule may be used advantageously for preliminary determinations of displacement and stability in the early stages of the design of a ship, and since extreme accuracy is not then necessary, it is usually possible to select stations on the design which are near enough to those required by the rule for preliminary calculations.

Correction of Ends.—Before applying the trapezoidal rule to ship calculations, computers frequently make certain end corrections which are most clearly explained by examples.

The simplest correction occurs when the water-line or station, though it ends at a regular ordinate, is of such form as to clearly lead to an error when the rule is applied directly.

In Fig. 7 a water-line is shown ending in a wide stem; the line

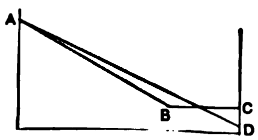


FIG. 7.

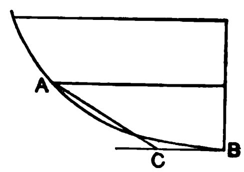


FIG. 8.

ABC is replaced by the straight line *AD*, which is drawn by the eye to include an equal area. Fig. 8 shows a round turn of the bilge extended to the line of the keel; the curve *AB* is replaced by the straight line *AC*.

If the water-line does not end at a station, we may first replace the actual figure near the end by a triangle, and construct an equivalent triangle which will give an ending at the desired station. In Fig. 9 the curved contour *BC* is replaced by the straight line *BD*; then drawing *BE* and *DG* parallel to it, the triangle *EBD* is replaced by the triangle *EBG*; the end ordinate is zero, and the next ordinate is *BF* measured to the side of the final triangle instead of the base-line.

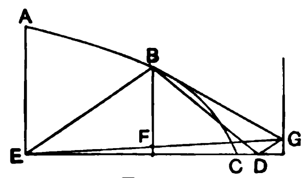


FIG. 9.

The corrections are comparatively small, consequently refinement is unnecessary and the constructions may be made rapidly.

It will be noted that the areas substituted are equal, as near as may be, to the real areas; but the moments of those areas about the base-line or about a transverse axis are likely to be different. Even if the differences of moments are appreciable, they are of less consequence than differences of areas, as will appear in the computations for displacement and stability.

The Navy Department uses the trapezoidal rule with ten water lines and twenty or more stations, as does also the French government. In England Simpson's rule is commonly used with seven or eight water-lines and about seventeen stations. It is probable that custom and prejudice have much influence with naval architects in their choice of a rule.

Planimeters and Integrators.—The use of planimeters and integrators makes such a reduction of the labor of determining displacement and stability of ships that they are indispensable, especially in designing ships. When skilfully used they give sufficient accuracy for all computations except the calculations of displacement and trim. There are several kinds of planimeters made, but of these Amsler's planimeter appears to be best adapted to ship-work; Amsler also makes an integrator on the same general principle which gives also moments and moments of inertia. Coradi makes a planimeter which is more accurate than the Amsler integrator, and also makes an integraph which will be described in connection with differential and integral curves.

Amsler's Planimeter.—The common form of Amsler's planimeter, known as the polar planimeter, consists of two arms hinged together, as represented by Fig. 10; one arm, called the guiding-arm, has a needle-point which is thrust into the paper upon which the figure to be measured is traced; the other arm, called the tracing-arm, carries the tracing-point and the measuring-wheel on a parallel axis. To measure the area of a figure the instrument is laid on the paper in a convenient position with the tracing-point at a known point on the contour of the figure, and with the needle-point held down by a weight placed over it. The scale on the wheel is read by aid of a vernier, then the tracing-point is carried around the contour of the figure in right-handed direction to the starting-point, after which

the wheel is read again. The difference of the readings multiplied by the proper factor gives the area of the figure.

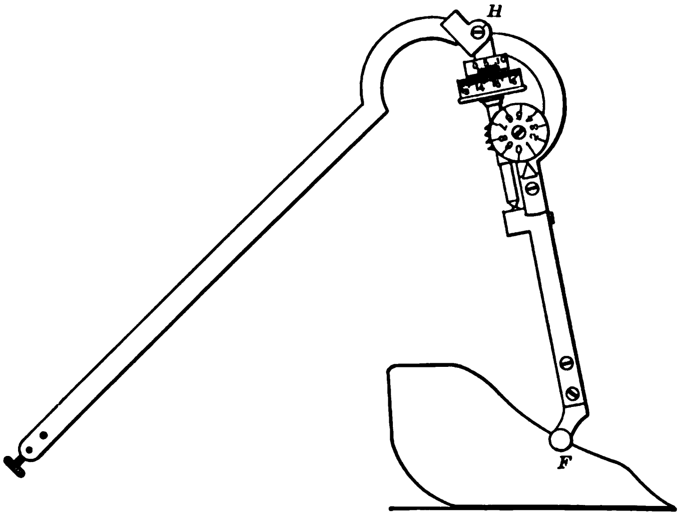


FIG. 10.

The following simple demonstration will show that the action of the instrument is correct, and will explain how to determine the constants for a given instrument.

Suppose the tracing-arm and its wheel, detached from the guiding-arm is moved parallel to itself as in Fig. 11, until the wheel has made one turn. The wheel will roll the distance *wv* equal to the altitude *hi* of the rectangle, of which the base is the length of the arm. The area of the rectangle is consequently equal to the length of the arm multiplied by the circumference of the wheel. Suppose the arm to be 4 inches long, and the wheel to have a circumference of 2.5 inches, that is, the wheel has a diameter of 0.7957 of an inch. Then the area of the rectangle will be 10 square inches and the factor for the instrument will be 10; some planimeters have an arm 8 inches long, and the factor is then 20; or the length

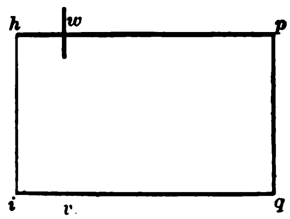


FIG. 11.

of the arm is sometimes adjustable to give various factors, so that areas may be measured in feet or on the metric system.

Suppose now that the arm is moved parallel to itself so that the tracing-point traces the arc of a circle from p to q; the wheel will roll a distance yx equal to the altitude of the figure, and will slide the distance wy, which, however, does not affect the reading of the wheel; the area of the figure will, of course, be properly measured.

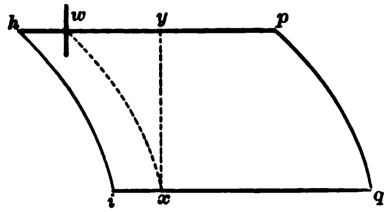


FIG. 12.

Suppose now that the instrument is properly connected and applied to measure the figure $pqrs$, starting at p, Fig. 13. The tracing-arm first moves parallel to itself from hp to iq, measuring the area of the parallel-sided figure $hpqi$. Let the arm now swing on the

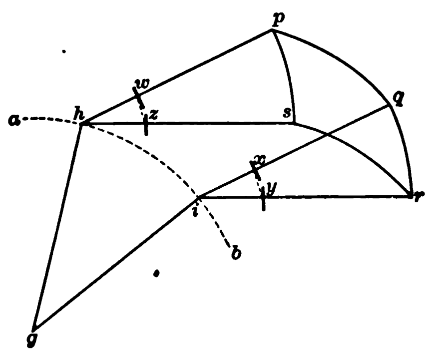


FIG. 13.

hinge as the point traces the arc qr; the wheel rolls over the arc xy and adds a corresponding amount to the record. Let the arm now move to hs, measuring the parallel-sided figure $hsri$, but in reverse direction so that its area is subtracted from the record on the wheel. Finally let the point trace the arc $sp = qr$ and in the reverse direction; the wheel will roll the arc $zw = xy$ and will subtract the corresponding record from the wheel, and consequently the total swinging of the arm forward and back through equal angles will have no influence on the reading of the wheel. But the area of the figure $pqrs$ may

be obtained by adding the parallel-sided figure *hpqi* and the sector *qir* and subtracting the parallel-sided figure *hsri* and the sector *phs*; consequently the planimeter correctly measures the area of the figure traced.

If an irregular contour is to be measured, as in Fig. 14, we may replace the figure by a number of figures like *pqrs*, Fig. 13, all of which can be correctly measured by the planimeter. Instead of tracing the figures individually, we may trace them successively, leaving out lines like *ec*, *ig*, etc., which would be traced twice in contrary directions, beginning and finishing at the same point. By increasing the number of quadrilaterals and diminishing their widths, the irregular contour may be made to approach as closely as we please to the smooth contour of the figure to be measured, and consequently by tracing the smooth contour itself we may correctly measure the included area.

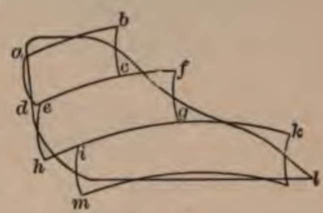


FIG. 14.

If a number of contiguous figures or a number of overlapping figures are to be measured by a planimeter, we may trace all the figures successively, beginning and ending at the same point, and reading the measuring-wheel at the beginning and the end; the difference of the readings multiplied by the proper constant gives the sums of the areas of all the figures. As stated in the description of the instrument, the tracing-point is carried around the figure in right-handed direction; if a figure is traced in left-handed direction, the final reading of the wheel is less than the initial reading, which is equivalent to subtracting the area of the figure. If some figures of a given combination of contiguous or overlapping figures are to be added and some are to be subtracted, the latter are to be traced in left-handed direction. If a given figure is too large to be traced by the planimeter, it may be broken up into a number of parts which may be measured separately; or the guiding-point may be placed near the centre of the figure and the contour may be traced by a continuous forward motion of the tracing-arm without any return or backward motion of that arm. In that case a certain constant, determined separately for each instrument and usually engraved upon

it, is to be added to the difference of the readings of the measuring-wheel at the beginning and end. Suppose, first, that we have to do with an irregular contour made up of arcs of circles like *pqrst*, Fig. 15, which can be traced by moving the tracing-arm parallel to itself and

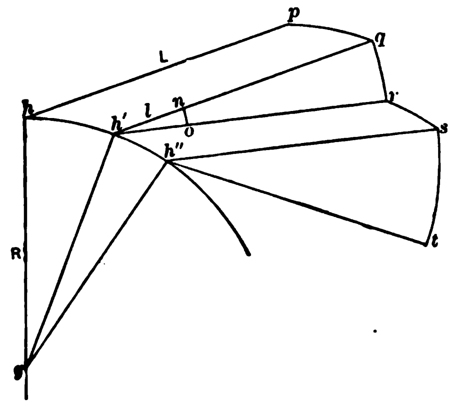


FIG. 15.

then swinging it through an angle. The parallel-sided figures like *pqh'h* are properly measured by the planimeter. When the arm swings through the angle *qh'r* the wheel rolls over an arc *no*, which has no relation to the area, and the record must be corrected by a corresponding amount. The tracing-point after tracing the entire contour of the figure returns to the starting-point, at which the tracing-arm occupies identically its original position. The sum of all the arcs like *no* will be the circumference of a circle with a radius equal to l, the distance of the measuring-wheel from the hinge. The corresponding area will be

$$2\pi l \,.\, L,$$

where L is the length of the tracing-arm. In addition to the sum of all the parallel-sided figures, the contour will enclose a series of sectors of a circle like *qh'r*, the sum of which will be equal to a circle having the area

$$\pi L^2;$$

and also the circle traced by the hinge, which has the area

$$\pi R^2.$$

The resultant area will be

$$\pi R^2 + \pi L^2 - 2\pi lL,$$

which must be divided by the instrumental factor to find the constant engraved on the instrument

Sometimes the measuring-wheel is carried by a prolongation of the tracing-arm beyond the hinge, and in this case the correction $2\pi lL$ is to be added.

If the constant for a planimeter is unknown, it may be determined by measuring a known circle with the guiding-point at the centre; a bridle or a strip of paper may be used to keep the tracing-point at a constant distance from the centre of the circle and aid in tracing its circumference. The method of passing from an irregular contour of circular arcs like *pqrst*, Fig. 15, to a smooth contour is the same as that for passing to the smooth contour of Fig. 14, and the

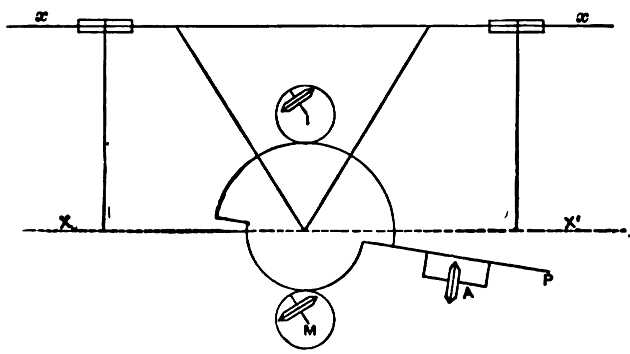


FIG. 16.

conclusion is likewise that the area of such a curve is properly measured when its contour is traced by the planimeter. In general it is likely that the circle traced by the hinge of the planimeter will be entirely within the contour of the figure to be measured, but the planimeter may be used just as well when the circle of the hinge is partly outside of the contour.

Amsler's Integrator.—Fig. 16 represents a mechanical integrator which, in addition to the area, measures the moment and moment of inertia of the figures traced by the tracing-point. The instrument is guided by a carriage which rolls in a straight groove in a steel bar; this bar may be set at the proper distance from the hinge of the

tracing-arm by aid of trams, as shown; the line XX' which passes through the points of the trams and under the hinge is the axis about which moments and moments of inertia are measured. The tracing-arm carries the usual measuring-wheel for areas, on an axis parallel to the tracing-arm; setting the arm to one side does not affect the rolling of the wheel when the area moves parallel to itself, and whatever effect on the wheel the swinging of the tracing-arm may have is counterbalanced by swinging that arm back, for the tracing-arm starts and stops at identically the same place. The tracing-arm carries two toothed sectors which gear with the two disks M and I, in such a manner that M turns through twice the angle to which the tracing-arm is swung from the axis XX', and I turns through three times that angle. These disks carry measuring-wheels which measure the moment and moment of inertia of the figure traced. When the tracing-point is on the axis XX' the axis of the area wheel is parallel to the line XX', as is also the axis of the moment-of-inertia wheel, while the axis of the moment wheel is perpendicular to that line.

The action of the area wheel is the same as that of the polar planimeter, the only difference in the construction of the instrument

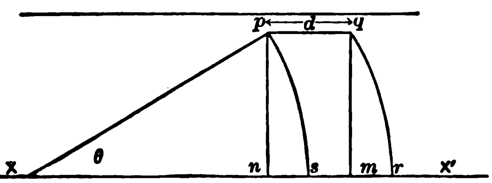


FIG. 17.

for this purpose being that the hinge is guided on a straight line instead of an arc of a circle.

The typical figure upon which the demonstrations are to be based is $pqrs$, Fig. 17, bounded by two arcs with a radius equal to the length of the tracing-arm and by a portion sr of axis XX' and a parallel and equal line pq. This figure has the same area, moment, and moment of inertia as the figure $pqmn$ made by dropping perpendiculars from p and q on to the line XX'. The area of the rectangle $pqmn$ and also of the quadrilateral $pqrs$ is equal to the product of the base $sr=d$ multiplied by the altitude, which for the angle θ of

the tracing-arm with the line XX' is $L \sin\theta$, where L is the length of the tracing-arm. The area of $pqrs$ is consequently

$$Ld \sin\theta.$$

When the tracing-point traces the line pq the area wheel slides and rolls over an equal and parallel line, but as its axis makes an angle of θ with its path it rolls the distance $d \sin\theta$, and consequently it records properly the area,

$$Ld \sin\theta. \qquad (1)$$

When the tracing-point moves from r to s the axis of the area wheel is parallel to that line and it slides without rolling. The effect of swinging the arm from q to r is compensated by swinging it back from s to p.

The moment of the rectangle $npqm$ and also of the quadrilateral $pqrs$ is equal to

$$dL \sin\theta \times \tfrac{1}{2}L \sin\theta = \tfrac{1}{2}L^2 d \sin^2\theta = \tfrac{1}{4}L^2 d(1-\cos 2\theta). \qquad (2)$$

The axis of the moment wheel is at right angles to the line XX', Fig. 17, when the tracing-point is on that line; and since the radius of the moment disk is half the radius of the circular rack with which it gears, its axis turns through the angle 2θ when the tracing-arm turns through the angle θ. The moment wheel rolls and slides the distance d while the tracing-point moves from p to q, but as its axis makes an angle of

$$90^\circ - 2\theta$$

with that line it rolls the distance

$$d \sin(90^\circ - 2\theta) = d \cos 2\theta;$$

the divisions of the moment wheel are so numbered that it appears to roll backward during this operation. When the tracing-point moves from r to s the moment wheel rolls the distance $d = sr = pq$ forwards, for its axis is then perpendicular to XX'. Whatever effect the swinging of the tracing-arm from q to r may have on the

moment wheel is compensated by swinging that arm from p to s. Consequently the net distance through which the wheel rolls is

$$d(1-\cos 2\theta),$$

which is to be multiplied by $\frac{1}{4}L^2$ to find the moment as indicated by equation (2).

The moment of inertia of the rectangle $pqmn$ and of the quadrilateral $pqrs$ is equal to

$$\begin{aligned} \tfrac{1}{3}dL^3 \sin^3\theta &= \tfrac{1}{3}L^3d \sin^3\theta = \tfrac{1}{3}L^3d(\tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta) \\ &= \tfrac{1}{4}L^3d \sin\theta - \tfrac{1}{12}L^3d \sin 3\theta. \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

When the tracing-point moves from p to q the moment-of-inertia wheel slides and rolls through the same distance which is represented by d; but its axis, which was parallel to the line XX' when the tracing-point was on that line, was turned through the angle 3θ when the tracing-arm moved through the angle θ, because the radius of the moment-of-inertia disk is one-third of the radius of the circular rack with which it gears. Consequently the wheel rolls the distance $d \sin 3\theta$, while the tracing-point moves from p to q. The moment-of-inertia wheel, like the area wheel, slides without rolling while the tracing-point moves from r to s. As before, with the area and moment wheels, the effect of swinging the tracing-arm from q to r is counterbalanced by swinging that arm from s to p. In order to get the moment of inertia of the figure $pqrs$ as expressed by the equation (3), it is now necessary to multiply the distance the moment-of-inertia wheel has rolled by $\frac{1}{12}L^3$, and to subtract that result from $\frac{1}{4}L^3$ multiplied into the distance the area wheel has rolled, namely, $d \sin\theta$.

Integrators of this type for English units may have the tracing-arm 8 inches long from hinge to tracing-point and the circumferences of the area and moment wheels are 2.5 inches, while the circumference of the moment-of-inertia wheel is 2.3438 inches. This last dimension is chosen to get a convenient factor for that wheel. Turning to equation (1), where L is the length of the arm and $d \sin\theta$ is the distance the area wheel has rolled, it is evident that when the latter is 2.5 inches the wheel will make one complete turn; conse-

quently the factor by which the readings of the area wheel are to be multiplied is

$$8 \times 2.5 = 20.$$

From equation (2) we find in like manner that the factor for the moment wheel is

$$\tfrac{1}{4} \times 8^2 \times 2.5 = 40.$$

Finally, from equation (3) it appears that to get the moment of inertia the reading of the area wheel is to be multiplied by the factor

$$\tfrac{1}{4} \times 8^3 \times 2.5 = 320,$$

and that the reading of the moment-of-inertia wheel is to be multiplied by

$$\tfrac{1}{12} \times 8^3 \times 2.3438 = 100,$$

and the result is to be subtracted from that for the area wheel. For this integrator we may have the following equations, if the difference of readings of the area wheel is $a_2 - a_1$, for the moment wheel $m_2 - m_1$, and for the moment-of-inertia wheel $i_2 - i_1$.

$$\text{Area} = 20(a_2 - a_1),$$

$$\text{Moment} = 40(m_2 - m_1),$$

$$\text{Moment of inertia} = 320(a_2 - a_1) - 100(i_2 - i_1).$$

To pass from the typical figure *pqrs*, Fig. 17, to a figure of any form for which the area, moment, and moment of inertia is desired, it is sufficient to consider that an approximation may be had by adding and subtracting a sufficient number of figures like the typical figure, some on one side and some on the other side of the axis XX', and that by reducing the width of these figures and increasing the number the approximation can be carried to any desired degree, and that, finally, the correct result can be had by tracing the contour of the desired figure itself.

Graphical Integration.—The forms and properties of ships are usually represented or expressed by curves referred to rectangular

coordinate axes; these curves have, therefore, forms that can be represented in a general manner by the equation

$$y = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

but the analytical form of the function is commonly not determined, and in many cases it is inconvenient or impossible to determine it; nevertheless it is convenient to consider that the curves are of the sort that are commonly represented by equation (1). For example, the curve *ABCD*, Fig. 18, is like a water-line of a short and bluff

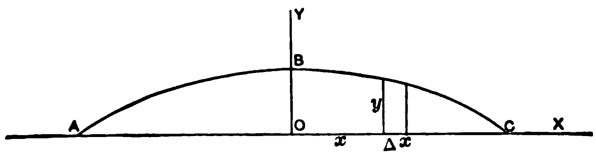


Fig. 18.

ship; it may be referred to the axes *OX*, *OY*, with the origin at the middle of the length. At any distance x from the origin the half-breadth is y, which can be measured at once from the drawing. If the curve *ABC* can be represented by an explicit equation, then y may be computed for any value of x. And further, if we desire to find the area we may proceed to integrate in the usual way; thus the elementary area is ydx and the entire area is

$$A = \int y dx = \int f(x) dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the curve is not represented by an equation, we may make an approximate determination by applying some rule of mensuration like Simpson's rule, for which purpose the length is divided into convenient intervals at which the half-breadths are measured. If we have a planimeter adapted for the purpose, we may use it to measure the area.

Suppose now that the moment of the figure *ABCD* is desired about the axis *OX*; then the moment of the elementary figure is $\frac{1}{2}y^2 \Delta x$, and the entire moment is

$$M = \int \tfrac{1}{2} y^2 dx = \int f_1(x) dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

There are now three methods of procedure open. (1) We may divide the length into convenient intervals, measure the half-breadths, compute the half-squares, and apply a mensuration rule like Simpson's rule to compute the moment approximately; or (2) we may compute the half-squares of the half-breadths of the water-line and plot

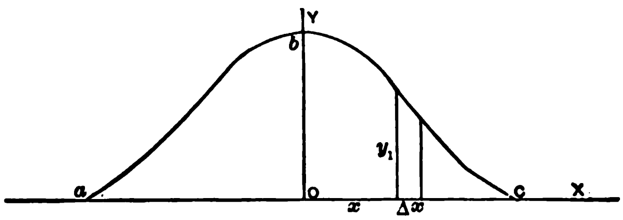


FIG. 19.

a new curve like Fig. 19, in which the ordinates like y_1 are each equal to the half-squares of the corresponding ordinates of Fig. 18. In this case equation (3) may be written

$$M=\int \tfrac{1}{2}y^2dx=\int y_1dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and it is evident that the area of the figure *abc* is numerically equal to the moment of the figure *ABC*, Fig. 18. If a planimeter adapted for the purpose is at hand we may measure the area of the figure *abc* by its aid and thus obtain the moment. Or (3) we may use a mechanical integrator, like Amsler's integrator, to measure the moment of the original curve.

If the moment of inertia of *ABC*, Fig. 18, with regard to the axis *OX* is desired, the elementary figure has the moment of inertia $\frac{1}{3}y^3\Delta x$, and the entire moment of inertia is

$$I=\int \tfrac{1}{3}y^3dx.$$

Here, again, we have three methods of procedure open: (1) we may compute the one-third cubes of the half-breadths and apply Simpson's rule or some equivalent rule, or (2) we may use the one-third cubes as ordinates of a new curve whose area shall be the desired moment of inertia, or (3) we may measure the moment of inertia of *ABC*, Fig. 18, by aid of Amsler's integrator.

In general, if the value of a function can be computed or measured at intervals, then the integral of that function can be determined by one of the first two methods explained for determining moments. That is to say, we may treat these values of the function as ordinates of a new curve and compute the area of that curve by some rule of mensuration such as Simpson's rule, or we may draw that curve and measure its area by a planimeter. This process is known as graphical integration, and is very useful when the form of the function is represented by a curve or when the integration cannot readily be performed.

Area of Angular Figures.—The area of a figure bounded by two straight lines and a curve such as AOB, Fig. 20, may most readily

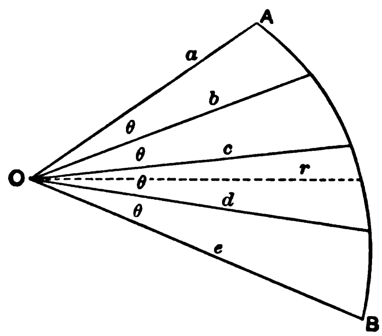


FIG. 20.

be obtained by referring it to polar coordinates, with the pole at O, the intersection of the two lines.

If r is any radius vector, then the elementary area is $\frac{1}{2}r \cdot r\Delta\theta$ and the total area is

$$A = \int \tfrac{1}{2}r^2 d\theta. \quad \ldots \ldots \ldots \quad (1)$$

To apply the method of graphical integration to this case we may treat the angular space θ as the abscissa of a curve referred to rectangular coordinates, and $\frac{1}{2}r^2$ as the ordinate, so that equation (1) may be transformed into

$$A = \int f(x)dx = \int y_1 dx. \quad \ldots \ldots \quad (2)$$

Commonly the computation is made by aid of either the trapezoidal

rule or Simpson's rule. As applied to Fig. 20, the trapezoidal rule gives

$$A=\theta\left(\frac{1}{2}\frac{a^2}{2}+\frac{b^2}{2}+\frac{c^2}{2}+\frac{d^2}{2}+\frac{1}{2}\frac{e^2}{2}\right),$$

and Simpson's rule gives

$$A=\frac{1}{3}\theta\left(\frac{a^2}{2}+4\frac{b^2}{2}+2\frac{c^2}{2}+4\frac{d^2}{2}+\frac{e^2}{2}\right).$$

The angle θ is to be expressed in circular measure; if given in degrees, it must be multiplied by the factor

$$\frac{\pi}{180}$$

to reduce it to circular measure.

Moments of Angular Figures.—The moment of a figure bounded by two straight lines and a curve about an axis through the vertex may be found by the following method:

In Fig. 21 let r be the radius vector of the curve AB at the angle α from the axis OT. As in the previous problem, the elementary

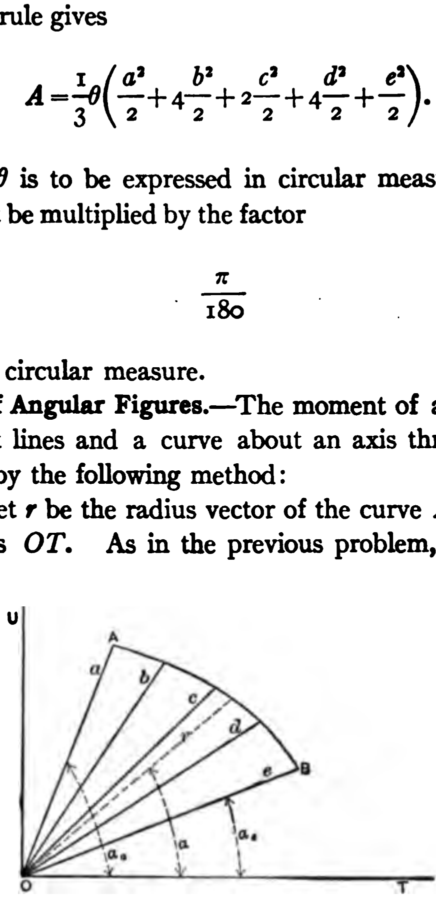


FIG. 21.

area is $\frac{1}{2}r \cdot r\Delta\theta$; the distance of its centre of gravity from OT is $\frac{2}{3}r \sin \alpha$; consequently the moment of the entire figure about OT is

$$\text{Moment}=\int\frac{r}{2}rd\theta\times\frac{2}{3}r\sin\alpha=\int\frac{r^3}{3}\sin\alpha d\theta. \quad . \quad . \quad . \quad (1)$$

Taking θ for the abscissa of a rectangular curve and $\frac{r^3}{3}\sin\alpha$ as the ordinate reduces this equation to the form

$$\text{Moment}=\int f(x)dx=\int y_2 dx. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the angular space be divided into equal intervals as in the preceding problem, the corresponding values of the function for the radii a, b, c, etc., will be

$$\tfrac{1}{3}a^3\sin\alpha_a,\quad \tfrac{1}{3}b^3\sin\alpha_b,\quad \tfrac{1}{3}c^3\sin\alpha_c,\quad \text{etc.}$$

For the trapezoidal rule the equation for the moment about ST is

$$\text{Moment}=\theta\left(\frac{1}{2}\frac{a^3}{3}\sin\alpha_a+\frac{b^3}{3}\sin\alpha_b+\frac{c^3}{3}\sin\alpha_c+\frac{d^3}{3}\sin\alpha_d+\frac{1}{2}\frac{e^3}{3}\sin\alpha_e\right),$$

and Simpson's rule gives

$$\text{Moment}=\frac{1}{3}\theta\left(\frac{a^3}{3}\sin\alpha_a+4\frac{b^3}{3}\sin\alpha_b+2\frac{c^3}{3}\sin\alpha_c+4\frac{d^3}{3}\sin\alpha_d+\frac{e^3}{3}\sin\alpha_e\right).$$

Here, as before, θ must be given in circular measure, and may be reduced by multiplying by the factor

$$\frac{\pi}{180}$$

if given in degrees; the various values of α will naturally be measured in degrees and the corresponding trigonometric functions taken from a table of natural sines.

The moment of the elementary area in Fig. 21 about the axis OU is

$$\tfrac{1}{2}r\,.\,rd\theta\times\tfrac{2}{3}r\cos\alpha,$$

and the moment of the entire figure is

$$\text{Moment}=\int\frac{r^3}{3}\cos\alpha d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the trapezoidal rule is used, we get

$$\text{Moment} = \theta\left(\frac{1}{2}\frac{a^3}{3}\cos\alpha_a + \frac{b^3}{3}\cos\alpha_b + \frac{c^3}{3}\cos\alpha_c + \frac{d^3}{3}\cos\alpha_d + \frac{1}{2}\frac{e^3}{3}\cos\alpha_e\right),$$

and Simpson's rule gives

$$\text{Moment} = \frac{1}{3}\theta\left(\frac{a^3}{3}\cos\alpha_a + 4\frac{b^3}{3}\cos\alpha_b + 2\frac{c^3}{3}\cos\alpha_c + 4\frac{d^3}{3}\cos\alpha_d + \frac{e^3}{3}\cos\alpha_e\right),$$

where it must be remembered that θ is the interval in circular measure; if the interval is given in degrees, it must be reduced by multiplying by $\pi \div 180$.

Differential and Integral Curve. — In Fig. 22 let ab be any given curve represented by the equation

$$y = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and let the curve AB be represented by the equation

$$y' = \int f(x)dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The second curve may be obtained graphically as follows: Measure or compute the area of the figure $Aann'$, and at n' erect the ordinate $n'N$ equal to that area; in like manner determine a sufficient number of points and draw a fair curve ANB: this curve is called the integral curve, and the original curve is called the differential curve; the ordinates of the differential curve may be expressed in linear units (feet), and then the ordinates of the integral curve will represent units of area (square feet). Differentiating equation (2),

$$y = f(x) = \frac{dy'}{dx}; \quad . \quad . \quad . \quad (3)$$

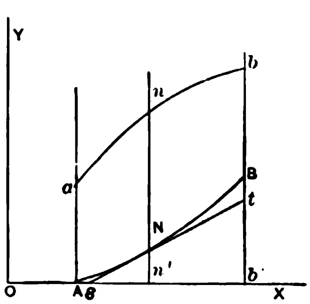


FIG. 22.

so that the ordinate of the differential curve is equal to the tangent of the angle which a tangent to the integral curve makes with the axis of abscissæ. To obtain y graphically, draw a tangent st at N;

then

$$\frac{Nn'}{sn'} = \tan Nsn' = \frac{dy'}{dx} = f(x) = y. \quad . \quad . \quad . \quad . \quad (4)$$

The construction of the differential curve from the integral curve is unsatisfactory because it is difficult to draw a tangent to a curve at a given point.

The following construction for the tangent at a point is correct for a parabola with its axis vertical, and will usually give a fair approximation when applied to other curves. At the required point A, Fig. 23, draw the ordinate Aa and lay off $ba=ac$; erect the ordinates bB and cC and draw the chord BC; a line Ad through A and parallel to BC will be the desired tangent. If the curve BAC is not a parabola, the equal distances ab and ac must be short and the line BC will also be short and cannot give a good determination of the direction of the tangent.

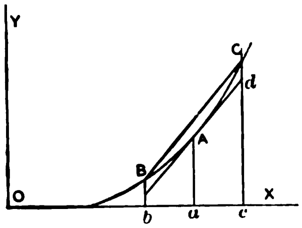


FIG. 23.

Integraph.—A recording integrator or integraph devised by Abdank-Abakanowicz and made by Coradi is represented by Fig. 24, which is a simplified diagram omitting minor details. The frame of the instrument is a rigid bar AB resting on wheels W and W' of exactly the same diameter, so that the instrument may roll back and forth, remaining perpendicular to an axis XX' at the middle of the bar. At D is a tracing-point which can be moved along a differential curve like ab. Fig. 22, and meanwhile the recording point I will draw an integral curve like AB; the axes for the integral curve are parallel to those for the differential curve, but the origins do not coincide.

The essential feature of the instrument is a sharp-edged wheel at i which can roll but cannot slip on the paper. By the construction of the instrument the plane of this wheel is held so that its trace ii' on the plane of the paper makes an angle $i'in$ which is proportional to the ordinate dc of the differential curve at the point D. Consequently the recording-point I moves instantaneously on the tangent to the integral curve, and therefore the integral curve has the proper relation to the differential curve.

The arm which carries the tracing-point D slides along the right-hand edge of the frame AB, and remains perpendicular to it; in like manner the arm which carries the recording-point I slides along the left-hand edge of the frame and remains perpendicular to it; the latter arm drops down at the left of the bar and extends

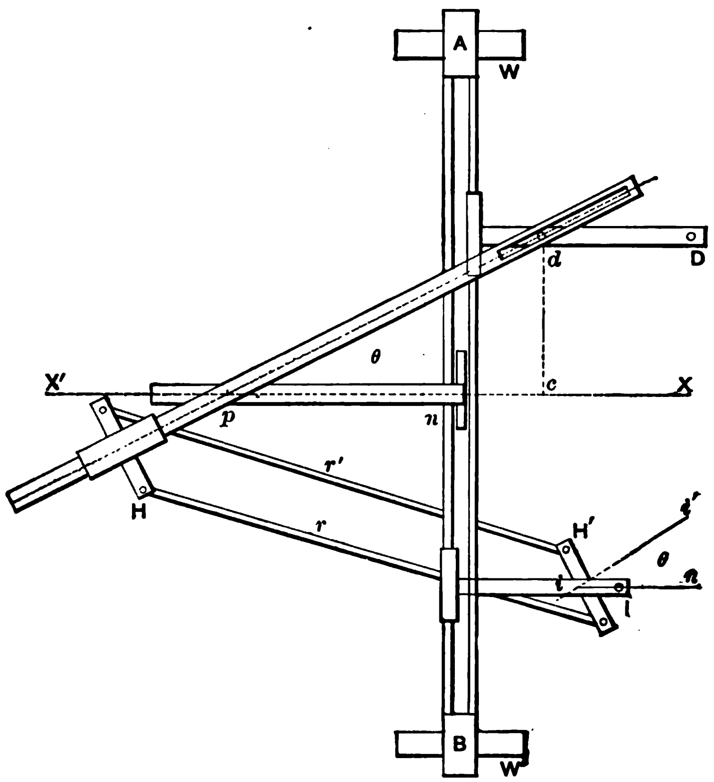


FIG. 24.

under it so that the two arms do not interfere; the recording-point I is always a constant distance to the left of the tracing-point D. At the middle of the frame is a fixed arm pn with a pivot at p, and on the arm dD is another pivot at d. The rod pd consequently makes an angle θ with the axis XX' which is proportional to the ordinate at D. The wheel at i has its axis held by a crosshead H' so that its plane is perpendicular to that crosshead, and the crosshead

H' is kept parallel to the crosshead H by the equal and parallel rods r and r'; this crosshead is fixed at right angles to a slide on the rod pd, and consequently the angle $i'in$ is always equal to θ as indicated.

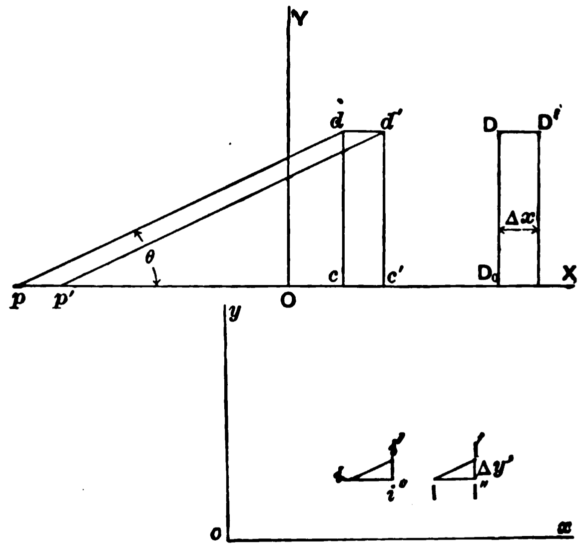


FIG. 25.

For simplicity of explanation let the tracing-point D, the recording-point I, the point of contact i of the wheel, and the pivots p and d be transferred to the diagram in Fig. 25. Let O be the origin for the differential curve, and o for the integral curve. If the tracing-point D is moved forward a small distance $\Delta x = DD'$ parallel to the axis OX, the wheel will roll the distance ii' and the recording-point will move from I to I' and will rise the distance

$$\Delta y' = I'I'' = \Delta x \tan \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

but the construction of the instrument gives

$$\tan \theta = \frac{dc}{pc} = \frac{1}{pc} y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that

$$y = pc \frac{\Delta y'}{\Delta x}$$

or at the limit as Δx decreases

$$y = pc\frac{dy'}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the essential relation of the integral and differential curves as indicated by the equation (3), p. 25, except that there is here introduced a factor which is the ratio of the scales for the ordinates of the two curves. The pivot d may be moved and set at any point on the arm dD so as to adjust the scales as desired. Thus if the scale for the differential curve is given in inches, and if the distance pc, Fig. 24, is made equal to four inches, then each inch of ordinate for the integral curve will represent four square inches. As a convenient verification of this statement suppose that pc is made four inches and that the ordinate DD_0 is also four inches, then if the point D is moved parallel to OX one inch, the increase of area will be four square inches. Now the angle θ under these conditions is $45°$ and one inch forward motion of the integraph will make the tracing-point rise one inch, which must correspond to four inches of area, and consequently the scale for ordinates of the integral curve is four square inches of area to an inch.

The edge of the frame on which the recording-arm slides is divided into inches, and the arm carries a vernier by aid of which the ordinates of the integral curve may be read directly and thus the integraph may be used to measure areas much as the planimeter does.

If the integraph is used for measuring large areas, the recording-arm is likely to approach the end A of the frame. In such case we may mark the position of D and take the reading for the recording-arm, then describe any figure or figures in left-hand rotation, ending at the marked position of the tracing-point; this will make the recording-arm approach the zero of the scale. We may now take a new initial reading for the tracing-arm and proceed to trace the curve from the marked point.

To adjust the integraph for use, the tracing-arm is clamped by a device for that purpose at the middle of the frame, which is then set by hand perpendicular to the axis XX' with the tracing-point on that line. The instrument is then adjusted by trial so that the tracing-

point shall remain on the axis when it is moved back and forth. The tracing-point has itself a slight adjustment parallel to the frame to facilitate the final setting. The instrument is balanced so as to have a small pressure on a foot near the tracing-point, sufficient to give stability. All the sliding parts have roller-bearings to reduce friction, and all the pivots are nicely made so that the instrument works with delicacy and precision.

CHAPTER II.

DISPLACEMENT AND CENTRE OF BUOYANCY.

THE displacement of a ship is the weight of water displaced by the ship when afloat. To determine the displacement the volume of water, in cubic feet, displaced by the ship is computed from the lines; this volume divided by the number of cubic feet of water per ton will give the displacement in tons. For this purpose it is customary to consider that 35 cubic feet of sea-water or 36 cubic feet of fresh water weigh one ton of 2240 pounds. At maximum density (39°.1 F.) one cubic foot of fresh water weighs about 62.425 pounds, so that 35.88 cubic feet weigh one ton at that temperature; at the same temperature one cubic foot of sea-water weighs about 64.05 pounds, and 34.97 cubic feet weigh one ton; at 66° F. the above assumptions are almost exactly right.

The normal displacement of a ship is computed under the assumptions (1) that it floats erect, (2) that it is on an even keel or else has the normal trim, and (3) that it has the normal draught, measured at the middle of the length. It is customary to calculate the displacement both for greater and for less draughts than the normal draught and to determine certain properties for these several draughts; but for all these calculations it is customary to assume the same trim. By trim is meant the difference between the draughts at the bow and at the stern; for example, it is customary to say that a ship is trimmed two feet by the stern if the draught there is two feet greater than at the bow. Calculations of displacement are sometimes made for a ship when inclined transversely and when given an abnormal trim, especially in the determination of stability; such calculations will be considered in connection with the discussion of stability in the next chapter.

The explanations of the calculations of displacement are most readily stated for a concrete example, and for this the U. S. Light

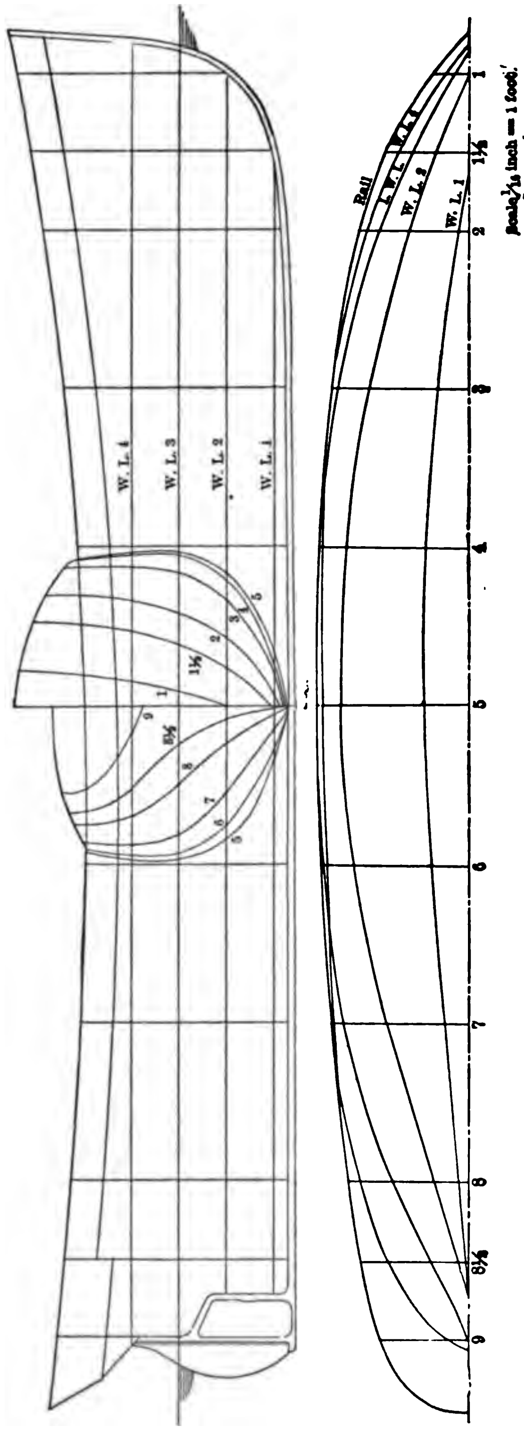

W. L. 4
W. L. 3
W. L. 2
W. L. 1
Rail
L. W. L.
W. L. 3
W. L. 2
W. L. 1
Scale 1/16 inch = 1 foot.
1
1½
2
3
4
5
6
7
8
8½
9

FIG. 26.

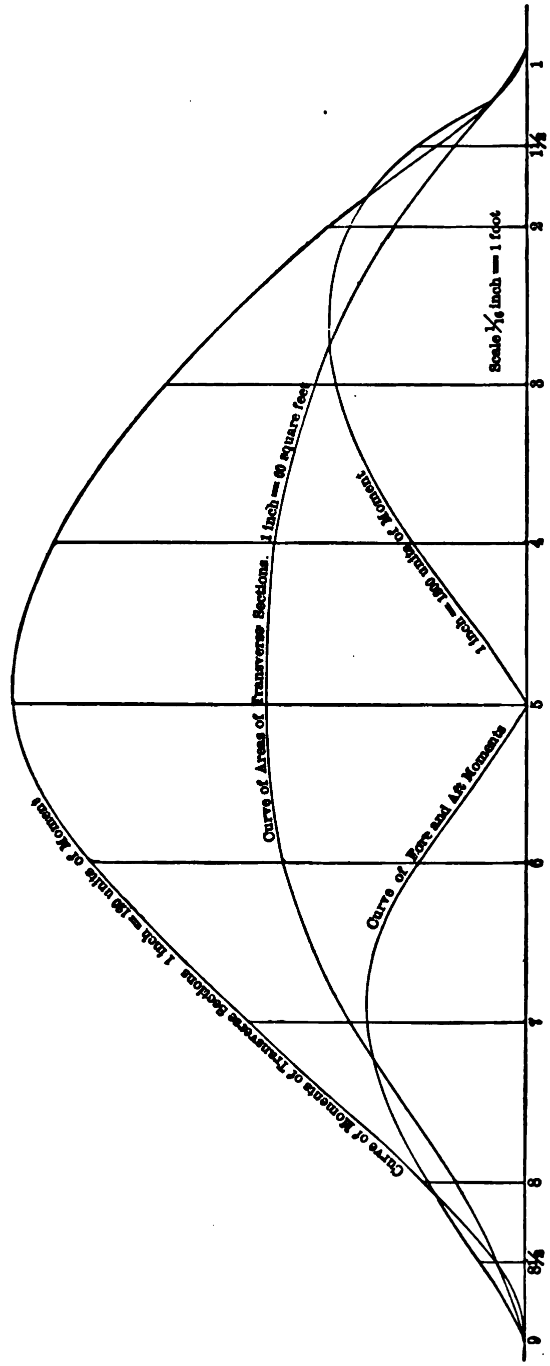

Curve of Areas of Transverse Sections. 1 inch = 60 square feet
Curve of Moments of Transverse Sections
Curve of Fore and Aft Moments
Scale 1/16 inch = 1 foot

FIG. 27.

Ship No. 51. has been chosen, especially as its form lends itself to calculations with only a few horizontal and vertical sections. The sheer plan, half-breadth plan, and body plans are given by Fig. 26. The principal dimensions are:

Length between perpendiculars	110 ft. 10 in.
Breadth, extreme moulded	26 ft. 6 in.
Breadth at water-line	26 ft. 3 in.
Draught, mean	9 ft.

The sheer plan and body plans show the load water-line and two other equidistant water-lines below, and also one water-line above the load water-line as straight lines. These water-lines and the rail are shown as curves on the half-breadth plan. On the other hand the vertical sections numbered 1 to 9 are shown as straight lines on the sheer plan and half-breadth plan, and as curves on the body plan. Half-stations numbered $1\frac{1}{2}$ and $8\frac{1}{2}$ are interpolated near the ends.

It should be noted that the word water-line is used with three meanings: (1) a line drawn on the sheer plan, (2) a horizontal plane through such a line, and (3) the figures cut from such a plane by the skin of a ship.

The end vertical sections or square stations are taken at or near the stem and the stern-post; these and the intermediate stations may or may not coincide with the frames of the ship. English naval architects commonly take seventeen stations numbered from the bow to the stern, and seven water-lines, one of which may be above the load water-line; the water-lines are usually spaced at even feet and there is a considerable portion of the ship, known as the main appendage, below the lowest water-line; the computation is made by Simpson's rule. French naval architects take twenty-one stations numbered each way from the midship section, and ten water-lines, dividing the draught at the midship section from the top of the keel to the water-line into ten equal parts; they use the trapezoidal rule. The custom of the Bureau of Construction and Repair of the U. S. Navy Department is to take from twenty to twenty-four stations and from ten to fourteen water-lines, of which two to four are above the normal load water-line; the trapezoidal rule is used.

The lines of a wooden ship always represent the skin or surface

outside of the planking, which is of considerable thickness. The lines of an iron or a steel ship represent the surface of the frames inside the plating. The displacement of the plating is calculated separately and added to the displacement of the main part of the hull together with that of the appendages, such as the keel, stem, stern-post, rudder, etc.

The most expeditious way of determining the displacement of a ship from the lines drawn to a reduced scale on paper is by aid of the Amsler planimeter. As this method is the simplest to explain, it will be given first. But as the displacement of a ship should be known with the greatest possible certainty and accuracy, the final determination should be made by numerical calculation from mould-loft dimensions, if possible. Forms for such numerical calculations, known as displacement sheets, will be explained later.

To determine the displacement from the lines of a ship the axis of the integrator is adjusted to the load water-line on the body plan (see Fig. 26), and the areas between the load water-line, the middle line of the body plan, and the several curves showing the form of the ship at the square stations are measured in succession. The integrator readings for areas, and the areas in square feet, are given in the following table:

AREAS OF TRANSVERSE SECTIONS.

Square Stations.	Initial Readings.	Final Readings.	Difference of Readings.	Areas, Sq. Ft.
1	0.0000	0.0003	0.0003	1.7
1½	0.0003	0.0046	0.0043	22.2
2	0.0046	0.0128	0.0082	42.1
3	0.0128	0.0258	0.0130	66.5
4	0.0258	0.0414	0.0156	79.6
5	0.0414	0.0575	0.0161	82.6
6	0.0575	0.0720	0.0145	77.9
7	0.0720	0.0830	0.0110	56.7
8	0.0830	0.0878	0.0048	24.5
8½	0.0878	0.0898	0.0020	10.2
9	0.0898	0.0898	0.0000	0.

NOTE.—The integrator here described can be read only to $\frac{1}{1000}$ of a revolution of the measuring-wheels; but for purpose of making a table of correct form an enlarged body plan was drawn from which readings were taken and correspondingly reduced.

Now one revolution of the area wheel of the integrator used for these measurements represents 20 square inches, so that the actual

areas of the half transverse sections could be obtained by multiplying the differences of readings by 20. But the scale of the drawing is $\frac{1}{16}$ of an inch to the foot, so that one linear inch represents 16 feet, and one square inch represents 16^2 square feet. Consequently the factor for reducing the differences of integrator readings to square feet is

$$20 \times 16^2 = 5120.$$

On Fig. 27 a base-line is drawn to represent the length of the ship, to the same scale as that of Fig. 26, and on it points are located to represent the square stations and numbered as before from 1 to 9 with half-stations at the ends. At each station the area of the half transverse section at that station is laid off as an ordinate to a convenient scale, and a fair curve, called *curve of areas of transverse sections*, is drawn through the ends of these ordinates. It is clear that the area of a half transverse section at any point intermediate between two square stations may be obtained by measuring the ordinate of the curves at that point. Again, we can get the approximate volume of a thin transverse slice of the displacement by multiplying the thickness of the slice measured along the longitudinal axis of the ship by the area at the middle of the slice. But the result thus obtained represents also the area of a narrow strip of the figure between the axis and the curve of areas on Fig. 27. By summing up the volumes of all the thin slices of displacement from stem to stern we get the approximate displacement of the ship, and by summing up the narrow strips of areas we get the area of the curve of transverse areas. A little consideration will show, consequently, that the area of the curve of transverse areas represents the volume of half the ship. The area can be readily measured by the aid of the integrator; in this case it is 6.12 square inches. Now the scale used in laying out the curve of areas was one inch equals 60 square feet, and the horizontal scale was, as before, one inch equals 16 feet. Consequently one square inch of the area under the curve of transverse areas represents

$$60 \times 16 = 960 \text{ cubic feet,}$$

and the volume of half the ship is

$$6.12 \times 960 = 5875 \text{ cubic feet.}$$

The displacement of the whole ship is

$$\frac{2\times 5875}{35}=335.7 \text{ tons.}$$

The *buoyancy* of a liquid on any body immersed in it (either wholly or partially) is the resultant upward pressure of the liquid on the wetted surface of that body. The resultant horizontal pressure is zero, or, in other words, the horizontal pressures are in equilibrium. The resultant vertical pressure is applied at the centre of figure of the body if wholly immersed; or if it is partially immersed, then at the centre of figure of the part immersed, that is, at the centre of the geometrical figure bounded by the wetted surface of the body and the continuation of the surface of the liquid through the body. The immersed part of a ship as described here is called the *carene.*

The *centre oj buoyancy* is the point of application of the buoyancy and is located at the centre of figure of the carene. It is customary to determine the centre of buoyancy at the same time that the calculation of the displacement is made. The importance of the determination of this point is at once seen when we consider that for equilibrium the centre of gravity of the ship (which may be controlled by the construction or loading of the ship) must be on a vertical line passing through the centre of buoyancy. All ships are symmetrical in form transversely, and are usually symmetrical in construction and loading, but they are not usually symmetrical fore and aft, either in form or construction and loading. The fore-and-aft location of the centre of buoyancy is required in calculations for trim, and the vertical location is required for calculations of stability.

Vertical Position of Centre of Buoyancy.—To determine the vertical position of the centre of buoyancy or the centre of figure of the carene, the moment of the carene is to be found with reference to the plane of the load water-line; the moment of the carene divided by the volume of the carene in cubic feet will give the distance of the centre of buoyancy below the load water-line. Consider a transverse slice of the carene one foot in thickness; its moment referred to the plane of the load water-line is numerically equal to the moment of the transverse section of the carene about its upper edge. The

moment of the half-section of the careen about its upper edge is readily obtained by aid of the integrator set to that line as an axis, and the determination of the moment of the half-carene then proceeds in much the same way as the determination of its volume. For this purpose adjust the axis of the integrator to coincide with the load water-line of the body plan of Fig. 26, and trace the several sections one after the other, taking readings of the moment wheel; these readings are conveniently taken at the same time as the readings for areas in the determination of the displacement, and the calculations are in practice carried through together. The following table gives the readings and moments:

MOMENTS OF TRANSVERSE SECTIONS.

Square Stations.	Initial Readings.	Final Readings.	Difference of Readings.	Moments.
1	0.0	0.00004	0.00004	6
1½	0.00004	0.00040	0.00036	61
2	0.00040	0.00121	0.00081	133
3	0.00121	0.00262	0.00141	231
4	0.00262	0.00447	0.00185	303
5	0.00447	0.00647	0.00200	328
6	0.00647	0.00795	0.00148	267
7	0.00795	0.00906	0.00111	182
8	0.00906	0.00947	0.00041	67
8½	0.00947	0.00958	0.00011	18
9	0.00958	0.00958	0.0	0

See note below table on page 35.

The constant for moments of the integrator used is 40, that is, one revolution of the wheel for moments is equivalent to the moment of 40 square inches at the end of an arm one inch long. But the scale of the drawing is $\frac{1}{16}$ of an inch to the foot, or one inch is equivalent to 16 feet. Consequently one square inch is equivalent to 16 square feet, and one inch of arm on the drawing is equivalent to 16 feet of arm. One revolution of the moment wheel is consequently equivalent in this calculation to

$$40 \times 16^2 \times 16 = 163840$$

units of moment in feet. The differences of readings in the preceding Table of Moments of Transverse Sections multiplied by 163840 give the moments of the vertical half-sections of the

ship about axes at their upper edges, as set down in the same table.

On Fig. 27 there is plotted a curve of *moments of transverse sections*, and just as the volume of the half-carene is obtained from the area of the curve of areas, so the moment of the half-carene can be obtained from this curve of moments. The scale for moments is 120 per inch of ordinate, and the scale of abscissæ is 16 feet per inch, so that the factor for transforming the area of the curve of moments is

$$120 \times 16 = 1920.$$

The area of the curve of moments measured by a planimeter is 10.6 square inches, consequently the moment of the half-carene with reference to the plane of the load water-line is

$$10.6 \times 1920 = 20352.$$

Dividing this moment by the volume of the half-carene gives for the distance of the centre of buoyancy below the load water-line

$$20352 \div 5875 = 3.47 \text{ feet}.$$

There is a practical advantage in drawing the curves of areas and of moments of transverse sections, since these curves should be fair, and important errors can consequently be detected.

After the areas and moments of the half transverse sections have been obtained by aid of the integrator, they may be treated by the trapezoidal rule or by some other rule for computing areas, to find the volume and moment of the carene, instead of the process here shown of drawing the curves and measuring their areas. Such a calculation is more expeditious than the method of drawing curves and integrating, and with a sufficient number of sections the results obtained may be as accurate as those obtained by aid of the integrator.

When great accuracy is not required the trapezoidal rule may be used for calculating the volume and displacement of the carene from the areas of the transverse sections with only a moderate number of stations. The end stations may be measured separately if they have an appreciable area, then all the intermediate stations may be

measured at one operation, running the tracing-point of the integrator around the several sections in succession without stopping to take intermediate readings; finally, the difference of the final and initial readings of the integrator will give the sum of the areas and the sum of the moments for all the sections traced. These sums of readings multiplied by the proper factors will give the sum of the areas and of the moments of the half transverse sections, which sums when multiplied by the distance between stations will give the volume and the moment of the half-carene from which the distance of the centre of buoyancy can be calculated. For the light-ship used in our previous work, if the end stations are neglected, the differences of integrator readings are

$$\text{for areas} \quad 0.0898 - 0.0003 = 0.0895;$$
$$\text{for moments} \quad 0.00958 - 0.00004 = 0.00954.$$

Now the scale of the drawing for Fig. 26 is 16 feet to the inch, and the integrator constants are 20 for areas and 40 for moments; further, the distance between stations for this ship is 13.43 feet, so that the computation for the distance of the centre of buoyancy below the load water-line gives

$$\frac{0.00954 \times 16^3 \times 40 \times 13.43}{0.0895 \times 16^2 \times 20 \times 13.43} = \frac{0.00954 \times 16 \times 2}{0.0895} = 3.4 \text{ ft.}$$

Another way of looking at the matter is to consider that the sums of the moments and the areas of the sections *in inches* are 20×0.0895 and 40×0.00954, so that the distance of the centre of figure below the axis is

$$2 \times 0.00954 \div 0.0895 = 0.213 \text{ inches,}$$

which is equivalent to 3.4 feet.

Fore-and-aft Position of Centre of Buoyancy.—If we had an integrator that could measure the moments of the several water-lines about a transverse axis at the middle of the length of the ship, we could get the fore-and-aft position of the centre of buoyancy by a method analogous to that just explained for finding the vertical position of the centre of buoyancy. But integrators are not made with reach enough for this purpose with drawings of convenient scale; it

is true that special lines can be drawn with diminished fore-and-aft scale and with the usual transverse scale, but the labor of drawing such a distorted set of lines would be too great to make such a method advisable.

The usual method is to calculate the fore-and-aft position of the centre of buoyancy from the transverse sections as follows. In the accompanying table the areas of the half transverse sections at the several square stations are taken from the table of transverse sections on page 35, and each section is multiplied by its distance from the fifth or midship section, thus giving the moments of the areas of the transverse half-sections about a transverse axis in the midship section. Moments forward of the midship section are considered to be positive, and moments aft are considered negative.

MOMENTS ABOUT MIDSHIP SECTION.

Stations.	Areas of Half-sections.	Distances from Midship Section.	Moments about Midship Section.
1	1.7	53.72	90
1½	22.2	47.00	1043
2	42.1	40.29	1696
3	66.5	26.86	1786
4	79.6	13.43	1069
5	82.6	0.00	0
6	77.9	13.43	1046
7	56.7	26.86	1523
8	24.5	40.29	986
8½	10.2	47.00	478
9	0.	53.72	0

The moments of the areas of the half transverse sections given in the last column of the preceding table are plotted as ordinates at the corresponding stations on Fig. 27, and a smooth curve through the ends of the ordinates is called the *curve of fore-and-aft moments.* Properly, the after-branch of the curve, which represents negative moments, should be below the axis, but for convenience it is plotted above the axis. The area of the forward part of the curve is 2.14 square inches, and the area of the after-part of the curve is 1.63 square inches, the difference being 0.51 square inch. The scale of the ordinates is 1800 square feet to the inch, and the horizontal scale is $\frac{1}{16}$ of an inch to the foot, consequently the factor by which

the area is to be multiplied to give the moment about an axis in the midship section is

$$1800 \times 16 = 28800.$$

The distance that the centre of buoyancy is forward of the midship section is obtained by dividing the product of the difference of areas and the factor just found by the volume of the half-carene, giving

$$0.51 \times 28800 \div 5875 = 2.5 \text{ feet.}$$

The determination of the fore-and-aft position of the centre of buoyancy by aid of a curve of fore-and-aft moments has been inserted to conform with our previous work and as an example of the method. The calculation by Simpson's rule is preferable for this case, and will furnish an example of the arrangement of work for that rule. Here, and elsewhere in this chapter, Simpson's rule is used instead of the trapezoidal rule or Tchebycheff's rule, as it appears suited for our present purpose; in practice the computer must determine which method or which rule will give the desired accuracy with the least effort.

In the following table the areas of the transverse sections are taken from the Table of Areas of Transverse Sections on page 35, and are multiplied by the Simpson's multipliers for the rule with half-spaces at the ends, giving in the fourth column a series of numbers which are called functions of areas. The sum of these functions multiplied by one-third the interval between stations gives the volume of the half-carene, thus:

$$\tfrac{1}{3} \times 13.43 \times 1303 = 5833 \text{ cu. ft.}$$

The discrepancy between this figure and the one found on page 36 is due to the crudity of the work on the light-ship, simplicity being sought at the expense of accuracy.

To get the moments of the half transverse sections about an axis in the midship section, we would naturally multiply the area of each half-section by its distance from the midship section, that is, by the distance between stations (13.43 ft.), and by the number of spaces the section is removed from the midship section.

CALCULATION OF FORE-AND-AFT POSITION OF CENTRE OF BUOYANCY.

Stations.	Half-areas.	Simpson's Multipliers.	Functions of Areas.	Arms.	Functions of Moments.	Sum of Functions.
1	1.7	½	0.8	4	3.2	
1½	22.2	2	44.4	3½	155.4	
2	42.1	1½	63.1	3	189.3	
3	66.5	4	266.0	2	532.0	
4	79.6	2	158.2	1	158.2	1038.1
5	82.6	4	330.4	0		
6	77.9	2	155.8	1	155.8	
7	56.7	4	226.8	2	453.6	
8	24.5	1½	36.7	3	110.1	
8½	10.0	2	21.2	3½	79.2	
9	0.0	½	0.0	4	0.0	798.7
			1303.4			239.4
						13.43
						3215.142
						2.46

The products thus obtained are then to be treated by Simpson's multipliers in order to apply Simpson's rule to the calculation of the moments of the volumes of the fore-and-aft parts of the carene. Thus at station 1 we should have

$$1.7 \times 4 \times 13.43 \times \tfrac{1}{2};$$

but in the table we have already

$$1.7 \times \tfrac{1}{2} = 0.8,$$

and for all the other stations we have the areas treated by the appropriate Simpson's multipliers. Reserving the distance between stations (13.43) to be used as a multiplier after summing up the functions of moments, we complete the column of functions of moments by multiplying the functions of areas by the number of spaces that each station is from the midship section. Summing up the functions of moments for the fore and the aft stations separately, and taking the difference, we have for the result 239.4 as given below the table.

The moment of the half-careen about an axis in the midship section is, therefore,

$$\tfrac{1}{3} \times 13.43 \times 239.4 \times 13.43,$$

and the volume of the half-carene is

$$\tfrac{1}{3} \times 13.43 \times 1303;$$

dropping factors that occur in both terms and dividing the moment by the volume, we have

$$\frac{239.4 \times 13.43}{1303} = 2.46 \text{ feet}$$

for the distance that the centre of buoyancy is forward of the midship section.

Curve of Areas of Water-lines. — The areas of the several water-lines are measured on the half-breadth plan in Fig. 26, by aid of the integrator, and recorded in the following table:

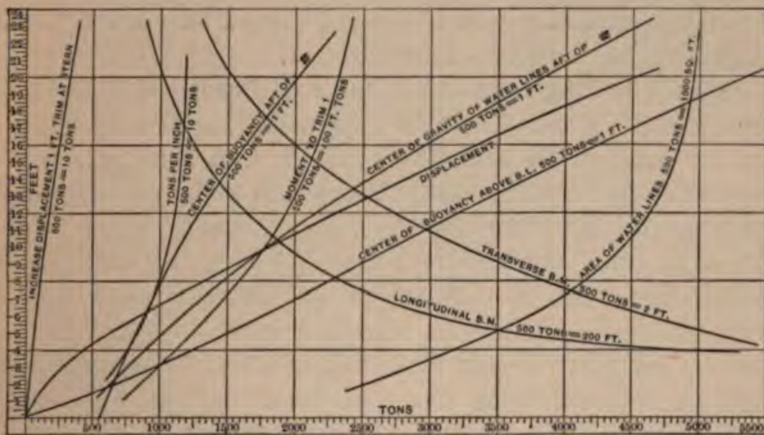
AREAS OF WATER-LINES.

Water-lines.	Areas, Square Feet.
1	475.5
2	1525.5
3	2106.8

On a diagram like Fig. 28 (which was constructed for another ship) a base-line is drawn to represent the top of the keel at the midship section, and the several water-lines are drawn (to scale); an axis of vertical coordinates is also drawn. On the several water-lines the areas of the water-lines, as recorded in the table above, are laid off to a convenient scale, and a fair curve is drawn through the points thus located. If there is a drag to the keel, the ***curve of areas of water-lines*** cuts the vertical axis below the base-line at a distance equal to the excess of draught of the stern-post over the draught at the midship section.

Curve of Displacement.—Just as the integration of the curve of areas of transverse sections gave the volume and displacement of the ship, so also we may get the volume and displacement by integrating the curve of areas of water-lines. The former curve is preferred for finding the displacement up to the load water-line, since there are more transverse sections than water-lines and the curve is more accurately located. On the other hand the curve of areas of water-lines may be integrated up to any water-line, and

thereby the displacement up to that water-line may be conveniently determined. On Fig. 28 the displacement up to the several water-lines is plotted and a fair curve is drawn through the points thus located, called the *curve of displacements.* This curve allows us to determine, by interpolation, the displacement up to any intermediate water-line, that is, the displacement for any draught, provided the trim of the ship is not changed. Moreover, it all w us to determine the increase of displacement accompanying an increase of draught; for we may, by interpolation, determine the displacement



ORDINATES = WATER LINES IN FEET.
ABSCISSAE = DISPLACEMENTS IN TONS.

FIG. 28.

at the original draught, and the difference of displacements is the increase due to the change of draught.

The curve of displacements is computed and drawn for salt water only; in practice it is customary to give also a curve for displacements in fresh water also; such a curve can be readily obtained by multiplying the abscissæ of the curve for salt water by $\frac{35}{36}$, the ratio of the cubic feet per ton for salt water to the corresponding quantity for fresh water.

Tons per Inch of Immerson.—For slight changes of draught it is convenient to calculate the increase of displacement by the following method: Multiply the area of each water-line by $\frac{1}{12}$ of one foot (one inch) and divide by 35; the result is approximately the

increase of displacement for one inch increase of draught, and may be plotted on the proper water-line on Fig. 28. A fair curve drawn through the points thus located, called the ***curve of tons per inch of immersion***, Fig. 28, allows us to find the increase of immersion per inch of increase of draught for any given draught, provided that the trim of the ship does not change. If the draught of a ship is changed only a few inches, the increase of displacement due to that change is obtained approximately by multiplying the change of draught in inches by the increase of displacement per inch of immersion for the mean draught. This approximate result gives better results for small changes of draught than can be had by interpolating on the curve of displacement.

Increase in Displacement for One Foot Change of Trim by the Stern.—All of the calculations for displacement, centres of buoyancy, etc., are based on the assumption that the trim is normal. If the vessel trims by the stern, the displacement at a given mean draught will be greater than if the ship were on an even keel. This is evident because the after-portion of the water-line is larger than the forward portion, and if the trim should change while the mean draught remained constant, the portion immersed would be greater than the part that emerges. So that if the trim of a vessel changes and the displacement remains the same, the mean draught will change. Since all of the curves of the displacement sheet are plotted on values of mean draught, if we must determine the displacement accurately it becomes necessary to make an allowance for the trim. In Fig. 29 let WL be the water-line with vessel on an even keel, and let $W'L'$ be the water-line with one foot trim by the stern. Let them intersect at O, the point half-way between draught-marks, which may or may not be at the midship section.

As the ship settles down by the stern there will be added to the displacement a wedge-shaped figure Oab and there will be taken away a wedge Odc. If the change of draught is not excessive, the area of the water-line $W'L'$ will not differ much from that of the water-line WL and we may compute the change of volume as though the wedges were segments of a body of revolution. Let the centre of gravity of the water-line WL be at the point g, slightly abaft the point O, then the gain of displacement in cubic feet will be

equal to the area of the water-line multiplied by the distance through which the point g moves. If the change of trim is one foot, then

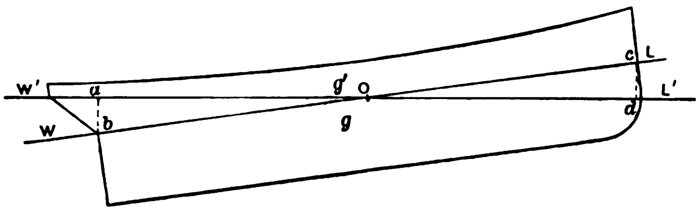


FIG. 29.

ab is half a foot, and so also is cd. If g' is the new position of g, we have for the approximate value of the motion of the centre of gravity

$$gg' = ab\frac{Og}{Ob} = \frac{1}{2}\frac{Og}{\frac{1}{2}L} = \frac{Og}{L},$$

where L is the length of the ship between the draught-marks and $a\ b$ is half a foot. The gain in volume will be

$$\text{Area of water-line} \times \frac{Og}{L},$$

and the increase in displacement will be

$$\text{Area of water-line} \times \frac{Og}{35L} \text{ tons.}$$

The curve called ***increase of displacement for 1 foot trim by the stern*** on Fig. 28 is determined in this way for each water-line.

Centres of Buoyancy.—It has already been pointed out that the integration of the area between the curve of areas of water-lines and the vertical axis on Fig. 28 gives the volume of the carene up to that water-line which is chosen for the vertical limit. This, of course, comes from the fact that the abscissa on a given water-line represents the area of that water-line, and the integration of the abscissæ gives the area of the curve, while the integration of the area of water-lines gives the volume of the carene. In like manner the moment of any abscissa about the base-line is numerically equal to the moment of the area of the corresponding water-line about the base-line, and the integration of the moments of the abscissæ about the base-line

gives the moment of the volume of the carene about that line. Consequently we may get the moment of the carene about the base-line by measuring with the integrator the moment of the curve of areas of water-lines about the base-line, the axis of the integrator being set on the base-line. The height of the centre of buoyancy above the base-line is readily obtained by dividing the moment of the carene about the base-line by the volume of the carene. For the sake of simplicity, as already pointed out, only a few stations and water-lines have been taken for this computation; the latter, in particular, are too few for a satisfactory location of the curves we have on Fig. 28; in practice these curves can be drawn with sufficient certainty if five points are located above the appendage. When there are enough water-lines the areas and moments of the curve of water-lines can be computed by Simpson's rule or the trapezoidal rule instead of measuring them by the integrator.

In Fig. 28 the distance of the centre of buoyancy above the base-line for the carene below each water-line is laid off as an abscissa on that line, thus locating points through which the *curve of centres of buoyancy* are drawn.

The centres of buoyancy are sometimes plotted from a reference-line drawn at 45° from the origin across the several water-lines (see Fig. 30). At the intersection of this line and each water-line an ordinate is drawn on which is located the centre of buoyancy of the carene below that water-line; the metacentre (to be discussed later) for the same carene is laid off on the same ordinate. A curve is drawn through the several centres of buoyancy called the *curve of the centres of buoyancy*. In using the curve of centres of buoyancy it is convenient to draw in a water-line (if necessary) at the desired draught and note its intersection with the 45° line; an ordinate is drawn at this intersection on which the distance of the centre of buoyancy below the water-line may be measured; at the same time the distance of the metacentre from the water-line or from the centre of buoyancy or from the centre of gravity of the ship may be measured on the same ordinate; the centre of gravity like the metacentre is reserved for future discussion, but it is instructive to note now the relation of the three points named.

Areas of Midship Sections.—On Fig 28 there is also a *curve*

of areas of midship sections obtained by plotting on each water-line the area of the midship section below that water-line.

The curve of areas of midship sections meets the vertical axis at the base-line, since that line is taken at the top of the keel at the midship section. Several of the curves on Fig. 28 meet the vertical axis at a point which, if there is a drag of keel, is below the base-line.

It will be observed that on Fig. 28 the displacement curve is plotted to some convenient scale, and that all the other curves are

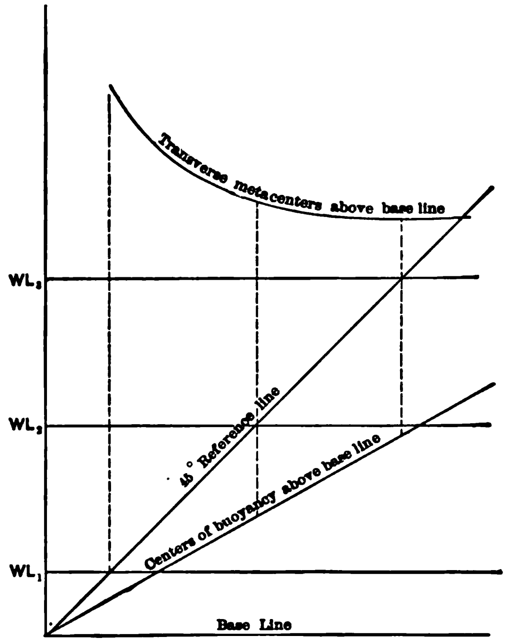


FIG. 30.

plotted in terms of this scale. The vertical scale is feet and the horizontal scale is tons. Thus the curve of centres of buoyancy may be laid off to such a scale that so many tons may be equal to one foot instead of so many inches to the foot; all curves laid off in this way may be read from a single scale instead of having a different scale for each curve.

The other curves on Fig. 28 will be explained in connection with the discussion of the properties represented by them.

DISPLACEMENT SHEET FOR U. S. LIGHT-SHIP NO. 51.

Distance between stations .... 13.43 ft.
" " water-lines...... 4.00 "

No. of Station.	Simpson's Mults.	Appendage below W. L. 1. Half-areas of Section.	Multiples of Area.	Leverage.	Functions of Moments.	Moments about B. L.	Functions of Moments.		Water-lines. 1		2		3		
									Simpson's Multipliers. 1		4		1		
1	½	.00	.00	4	.00	.00	.00		0.0	0.0	0.0	0.0	1.3	0.6	
									0.0				1.3		
1½	2	.00	.00	3½	.00	.00	.00		0.0	0.0	2.8	5.6	5.1	10.2	
									0.0		11.2		5.1		
2	3/2	.25	.37	3	1.11	.06	.09		1.0	1.5	5.3	7.9	7.9	11.8	
									1.0		21.2		7.9		
3	4	1.18	4.72	2	9.44	.51	2.04		2.8	11.2	8.8	35.2	11.5	46.0	
									2.8		35.2		11.5		
4	2	1.98	3.96	1	3.96	.98	1.96		3.8	7.6	10.7	21.4	13.0	26.0	
					14.51				3.8		42.8		13.0		
5	4	2.31	9.24	0	.00	1.08	4.32		4.2	16.8	11.3	45.2	13.2	52.8	
									4.2		45.2		13.2		
6	2	1.98	3.96	1	3.96	.99	1.98		3.4	6.8	10.0	20.0	12.6	25.2	
									3.4		40.0		12.6		
7	4	1.23	4.92	2	9.84	.66	2.64		2.2	8.8	7.2	28.8	10.8	43.2	
									2.2		28.8		10.8		
8	3/2	.49	.73	3	2.99	.28	.42		0.8	1.2	3.0	4.5	6.5	9.7	
									0.8		12.0		6.5		
8½	2	.16	.32	3½	1.12	.09	.18		0.3	0.6	0.9	1.8	3.2	6.4	
									0.3		3.6		3.2		
9	½	.00	.00	4	.00	.00	.00		0.0	0.0	0.0	0.0	1.7	0.8	
									0.0		0.0		1.7		
								Funct. Areas..		54.5		170.4		232.7	
								Multipliers....		1		4		1	
								Mults. of Areas		54.5		681.6		232.7	
								Leverage......		2		1		0	
								Funct. of M'ts.		109.0		681.6		0.00	790.6

```
       28.22        17.11          )13.63
       13.43        14.51  ------>0.47 ft.
    3)378.09         2.60  -->C.B. of Ap-
      126.33        13.43     pendage
           2      )34.92      below 1.
   35)252.66         1.24  -->C.B. of
Disp't of  }                  appen-
Appendage  } 7.22 tons.       dage aft
                              of 5
```

DISPLACEMENT SHEET.—*Continued.*

Vertical Sections.

Functions of Areas.	Multipliers.	Multiples of Areas.	Leverage.	Functions of Moments.
1.3	½	0.6	4	2.4
16.3	2	32.6	3½	114.1
30.1	1½	45.1	3	135.3
49.5	4	198.0	2	396.0
59.6	2	119.2	1	119.2
				767.0
62.6	4	250.4	0	0.0
56.0	2	112.0	1	112.0
41.8	4	167.2	2	334.4
19.3	1½	28.9	3	86.7
7.1	2	14.2	3½	49.7
1.7	½	0.8	4	3.2

969.0
(a) 11.94
35)11569.86
→330.56
→Tons disp't of main part.

586.0
767.0
181.0
13.43
969)2430.86
→2.51
→C.B. of main part for'd of 5.

Metacentres.

No. of Section.	Multipliers.	Ordinates of L. W. L.	Transverse. Cubes of Ordinates.	Transverse. Functions of Cubes.	Transverse. Functions of Ord's.	Longitudinal. Multipl'rs.	Longitudinal. Functions for C.G. of L. W. L.	Longitudinal. Multipl'rs.	Longitudinal. Functions for I. of L. W. L.
1	½	1.3	2.20	1.10	0.6	4	2.4	4	9.6
1½	2	5.1	132.65	265.30	10.2	3½	35.7	3½	124.9
2	1½	7.9	493.04	739.56	11.8	3	35.4	3	106.2
3	4	11.5	1520.87	6083.48	46.0	2	92.0	2	184.0
4	2	13.0	2197.00	4394.00	26.0	1	26.0	1	26.0
5	4	13.2	2299.97	9199.88	52.8	0	191.5	0	0.0
6	2	12.6	2000.87	4000.74	25.2	1	25.2	1	25.2
7	4	10.8	1259.71	5038.84	43.2	2	86.4	2	172.8
8	1½	6.5	274.62	411.93	9.7	3	29.1	3	87.3
8½	2	3.2	32.77	65.54	6.4	3½	22.4	3½	78.4
9	½	1.7	4.91	2.45	0.8	4	3.2	4	12.8

341.65 30202.82 282.7 166.3 827.2
35 (b) 2.98 191.5 1614.9
11957.75←→)90004.4 25.2 1335845
→7.52′ 13.43
)338.44
→Transverse metacentre above C.B. 1.45′ C.G. of L.W.L. for'd of 5

790.6
4
969.)3162.4
3.27′ C.B. of main part below L.W.L.

232.7
13.43
3)3125.16
1041.72
2
2083.44 sq. ft. area L.W L.

1335845.
4379
11957.75)1331466
111.35 ft.
Long metacentre above C.B.

$$a=\frac{13.43\times4\times2}{3\times3}$$

$$b=\frac{13.43\times2}{3\times3}$$

$$1614.9=\frac{\overline{13.43}^2\times13.43\times2}{3}$$

$$4.379=2083.4\times\overline{1.45}^2$$

DISPLACEMENT SHEET.—*Concluded.*

SUMMARY.

Calculation for Fore-and-aft Position of Centre of Buoyancy. Items.	Disp't. Tons.	Levers. Aft.	Levers. For'd.	Moments. Aft.	Moments. For'd.	Calculation for Vertical Position Centre of Buoyancy. Levers below L. W. L.	Moments below L. W. L.
Main Portion	330.56		2.51		829.7	3.27	1080.93
Appendage	7.22	1.24		8.95		9.24	66.71
Stem	.05		50.25		2.51	6.00	.30
Keel	.21	2.75		.57		13.48	2.83
Stern-post, etc.	.08	52.50		4.20		5.95	.48
Rudder	.12	55.00		6.60		4.35	.52
Propeller	.09	52.00		4.68		5.90	.53
Plating	3.32		.77		2.56	4.50	14.94

Tons. Total Dispt. 341.65 ← 25.00 834.7 341.65)1167.24

25.00 C. of B. below 3.42

→)809.77 L.W.L.

2 37 C. of B. for'd of 5.

RESULTS.

Total displacement	341.65 tons.
Centre of buoyancy below load water-line	3.42 ft.
Centre of buoyancy forward of No. 5	2.37 "
Transverse metacentre above centre of buoyancy	7.52 "
Longitudinal metacentre above centre of buoyancy	111.35 "
Area load water-line	2083.44 sq. ft.

Displacement Sheet.—The determination of the displacement of a ship is one of the most important operations connected with its design. For a preliminary design rapid methods of determining both displacement and stability are required to avoid delay and to permit of several recomputations if necessary; for the final design the displacement should be determined with accuracy and certainty, and for this purpose it is customary to make a numerical calculation from the mould-loft dimensions, if possible. As the computation is long and involved, it is necessary and customary to arrange it in some carefully prepared form, and this form is known as the *displacement sheet.*

The standard form of English displacement sheet will first be described as applied to the U. S. Light-ship No. 51, with only a few stations and water-lines; afterwards modifications of this form and also other forms will be discussed.

The main body of the ship, between the first and last square

station and between the load water-line and the first water-line above the keel, is calculated in the body of the displacement sheet, with the columns headed *Water-lines* and *Vertical Sections.* Subsidiary calculations for the main appendage are given at the left of page 50 and on page 51 for the metacentric height; the computation for the distance of the centre of buoyancy of the main part is transferred to the space below the calculations for metacentres. A summary of calculations and statement of results is given on page 52. In practice all the work of the displacement sheet is assembled on one sheet of sufficient size. In consequence of the small number of stations and water-lines the body of the displacement sheet is disproportionately small. That part of the ship which is below the first water-line is called the main appendage, and is computed on the left of page 50. Other appendages, like the projection of the bow forward of the main body, shell-plating, the stern-post, the keel, the rudder, the rudder-post, and the propeller (if there is any) are all treated as separate appendages; the calculations are not given in detail, but the volumes are set down in the table headed *Summary* on page 52. The lines for a wooden ship are taken to the outside of the planking, and there is consequently no separate calculation for the skin of the ship.

The displacement sheet is ruled in double columns for the water-lines, and in double lines for the square stations. The columns are numbered at their heads, under the title *Water-lines*, and the lines are numbered at the extreme left, under the title *No. of Station.* The half-breadths of the vertical sections or square stations are measured on the several water-lines, using the proper scale, and the results are set down in the left half of the column for water-lines, and in the top line of the double lines for stations; for example, the half-breadths of the fifth square station are 4.2 ft., 11.3 ft., and 13.2 ft. These half-breadths are also the half-breadths of the several water-lines at the several stations, and might have been measured on the half-breadth plan. For example, the half-breadths of the second water-line are 0.0, 2.8, 5.3, 8.8, 10.7, 11.3, 10, 7.2, 3.0, 0.9, and 00.

Having now the half-breadths of the several square stations, measured at regular vertical intervals, we may calculate the area of

each square station by Simpson's rule. The areas thus obtained may be treated by Simpson's rule, and thereby the volume of the carene may be determined. The formulæ for this work are as follows:

Sta.	Formulæ.		Areas.
1	$\frac{1}{3}\times 4(1\times 0.0+4\times 0.0+1\times 1.3)$	=	$\frac{1}{3}\times 4\times 1.3$
$1\frac{1}{2}$	$\frac{1}{3}\times 4(1\times 0.0+4\times 2.8+1\times 5.1$	=	$\frac{1}{3}\times 4\times 16.3$
2	$\frac{1}{3}\times 4(1\times 1.0+4\times 5.3+1\times 7.9)$	=	$\frac{1}{3}\times 4\times 30.1$
3	$\frac{1}{3}\times 4(1\times 2.8+4\times 8.8+1\times 11.5)$	=	$\frac{1}{3}\times 4\times 49.5$
4	$\frac{1}{3}\times 4(1\times 3.8+4\times 10.7+1\times 13.0)$	=	$\frac{1}{3}\times 4\times 59.6$
5	$\frac{1}{3}\times 4(1\times 4.2+4\times 11.3+1\times 13.2)$	=	$\frac{1}{3}\times 4\times 62.6$
6	$\frac{1}{3}\times 4(1\times 3.4+4\times 10.0+1\times 12.6)$	=	$\frac{1}{3}\times 4\times 56.0$
7	$\frac{1}{3}\times 4(1\times 2.2+4\times 7.2+1\times 10.8)$	=	$\frac{1}{3}\times 4\times 41.8$
8	$\frac{1}{3}\times 4(1\times 0.8+4\times 3.0+1\times 6.5)$	=	$\frac{1}{3}\times 4\times 19.3$
$8\frac{1}{2}$	$\frac{1}{3}\times 4(1\times 0.3+4\times 0.9+1\times 3.2)$	=	$\frac{1}{3}\times 4\times 7.1$
9	$\frac{1}{3}\times 4(1\times 0.0+4\times 0.0+1\times 1.7)$	=	$\frac{1}{3}\times 4\times 1.7$

In the table the Simpson's multipliers are set down at the head of the columns for water-lines, and the products of the half-breadths are set down immediately below the half-breadths; thus at the fifth station we have the products 4.2, 45.2, and 13.2. The sums of the products are set down in a column at the right, headed *Functions of Areas.*

The areas of the half vertical sections may now be treated by Simpson's rule to get the volume of the half-carene. It is convenient here to reserve the common factor $\frac{1}{3}\times 4$ and place it outside the parenthesis together with the factor for the new formula, giving (with half-stations at the end)

$$\frac{1}{3}\times 4\times \frac{1}{3}\times 13.43\{\tfrac{1}{2}\times 1.3+2\times 16.3+\tfrac{3}{2}\times 30.1+4\times 49.5$$
$$+2\times 59.6+4\times 62.6+2\times 56.0+4\times 41.8+\tfrac{3}{2}\times 19.3+2\times 7.1$$
$$+\tfrac{1}{2}\times 1.7\}=\tfrac{1}{3}\times 4\times \tfrac{1}{3}\times 13.43\times 969\ 0=5784.93 \text{ cu. ft.}$$

for the volume of the half-carene.

In the table the Simpson's multipliers are set down in a column to the right of the functions of areas, and the products are set down in another column headed *Multiples of Areas.*

The sum of this last column, 969.0, is multiplied by the continued product

$$2\times \tfrac{1}{3}\times 4\times \tfrac{1}{3}\times 13.43=11.94,$$

giving for the volume of the whole carene

$$11.94 \times 969.0 = 11569.86 \text{ cu. ft.}$$

The volume of the carene may also be calculated by finding the areas of the water-lines and then from those areas finding the volume by Simpson's rule.

The formulæ for this work are as follows:

First water line:

$$\begin{aligned}&\tfrac{1}{3} \times 13.43(\tfrac{1}{2} \times 0.0 + 2 \times 0.0 + \tfrac{3}{2} \times 1.0 + 4 \times 2.8 + 2 \times 3.8 \\ &\quad + 4 \times 4.2 + 2 \times 3.4 + 4 \times 2.2 + \tfrac{3}{2} \times 0.8 + 2 \times 0.3 + \tfrac{1}{2} \times 0.0) \\ &\quad = \tfrac{1}{3} \times 13.43 \times 54.5\end{aligned}$$

Second water line:

$$\begin{aligned}&\tfrac{1}{3} \times 13.43(\tfrac{1}{2} \times 0.0 + 2 \times 2.8 + \tfrac{3}{2} \times 5.3 + 4 \times 8.8 + 2 \times 10.7 \\ &\quad + 4 \times 11.3 + 2 \times 10.0 + 4 \times 7.2 + \tfrac{3}{2} \times 3.0 + 2 \times 0.9 + \tfrac{1}{2} \times 0.0 \\ &\quad = \tfrac{1}{3} \times 13.43 \times 170.4\end{aligned}$$

Third water line:

$$\begin{aligned}&\tfrac{1}{3} \times 13.43(\tfrac{1}{2} \times 1.3 + 2 \times 5.1 + \tfrac{3}{2} \times 7.9 + 4 \times 11.5 + 2 \times 13.0 \\ &\quad + 4 \times 13.2 + 2 \times 12.6 + 4 \times 10.8 + \tfrac{3}{2} \times 6.5 + 2 \times 3.2 + \tfrac{1}{2} \times 1.7) \\ &\quad = \tfrac{1}{3} \times 13.43 \times 232.7\end{aligned}$$

In the displacement table the products resulting from multiplying the half-breadths of the water-lines by the Simpson's multipliers are set down at the side of the half-breadths; and the sums of the products are set down in a line below the body of the table, with the title *Functions of Areas* at the side.

The areas of the water-lines may now be treated by Simpson's rule to calculate the volume of the half-carene, reserving the common factor $\frac{1}{3} \times 13.43$. The work is as follows:

$$\begin{aligned}&\tfrac{1}{3} \times 13.43 \times \tfrac{1}{3} \times 4(1 \times 54.5 + 4 \times 170.4 + 1 \times 232.7) \\ &\qquad = \tfrac{1}{3} \times 13.43 \times \tfrac{1}{3} \times 4 \times 969.0 = 5784.93.\end{aligned}$$

In the table the Simpson's multipliers are set down in a line below the functions of areas, and the products are set down in another line with the title *Multiples of Areas*. The sum of the multiples of areas is 969.0, the same as the multiples of areas for the transverse

sections, and is multiplied by the same factor,

$$2 \times \tfrac{1}{3} \times 13.43 \times \tfrac{1}{3} \times 4,$$

though the factors are arranged in a different order. The volume and displacement of the carene are, of course, the same for the two calculations and form a valuable check.

The volume of the carene divided by 35 gives the displacement of the ship in tons.

Centre of Buoyancy of the Main Body.—The determination of the position of the centre of buoyancy is similar to the calculation of the fore-and-aft position of the centre of buoyancy on page 41. Comparing with that table, we have in the displacement sheet the column of functions of areas, instead of the column of half-areas of sections. These same functions of areas could be multiplied by the factor $\frac{1}{3} \times 4$ and give us areas in square feet; but it is more convenient to reserve this factor. To get the moments of the areas of the transverse sections about an axis in the midship section, we should multiply the area of each section by its distance from the midship section; instead we shall multiply the functions of areas by the number of spaces that the section is from the midship section, having consequently in reserve the factor

$$\tfrac{1}{3} \times 4 \times 13.43.$$

These distances are set down in a column headed *Leverages*, and are multiplied into the multiples of areas, which have already been obtained by multiplying the functions of areas by the proper Simpson's multipliers. The resulting *Functions of Moments* form the parenthesis of a Simpson's formula which has the factor

$$\tfrac{1}{3} \times 13.43$$

before the parenthesis; and we have also in reserve the factor

$$\tfrac{1}{3} \times 4 \times 13.43,$$

giving in all the factor

$$\tfrac{1}{3} \times 4 \times 13.43 \times \tfrac{1}{3} \times 13.43.$$

But we should calculate the moments of the fore body and the after

body separately and take the difference, to find the moment of the half-carene about an axis in the midship section. We will consequently sum up the functions of moments for the fore body and the after body separately, getting

$$767.0-586.0=181$$

for the difference of the sums. The moment of the half-carene is therefore

$$\tfrac{1}{3}\times4\times13.43\times\tfrac{1}{3}\times13.43\times181.$$

But the volume of the half-carene is (see page 54)

$$\tfrac{1}{3}\times4\times\tfrac{1}{3}\times13.43\times969;$$

consequently the distance of the centre of buoyancy forward of the midship section is

$$\frac{\frac{1}{3}\times4\times13.43\times\frac{1}{3}\times13.43\times181}{\frac{1}{3}\times4\times\frac{1}{3}\times13.43\times969}=\frac{13.43\times181}{969}=2.5 \text{ ft.},$$

as set down in the displacement table.

Following a similar method the distance of the centre of buoyancy of the main body below the load water-line is found by multiplying the functions of areas at the bottom of the main part of the table by the number of intervals between water-lines for leverages and summing up the resultant functions of moments. The sum of vertical moments is then multiplied by

$$\frac{\frac{1}{3}\times13.4\times4\times\frac{1}{3}\times4}{\frac{1}{3}\times13.4\times\frac{1}{3}\times4}=4$$

and divided by 969, giving for the result

$$\frac{790.6\times4}{969}=3.27 \text{ feet}$$

for the distance of the centre of buoyancy of the main body below the load water-line.

Main Appendage.—To make the calculation for the main appendage, we replace the small parts of the transverse sections, cut off by the first water-line, by rectangles and triangles for which we may readily find the areas and centres of gravity. These areas

are set down in the column headed *Half-areas of Sections* and are treated by Simpson's multipliers, giving the ***Multiples of Areas.*** The sum of the multiples of areas multiplied by $\frac{1}{3} \times 13.43$ gives the volume of half the main appendage, from which the volume and displacement of the whole appendage is readily found.

The multiples of areas are now multiplied by the leverages, and the resultant functions of moments are summed up separately fore and aft, and the difference of the sums is multiplied by the distance between stations and divided by the half-volume of the main appendage to find the distance its centre of buoyancy is abaft the midship section.

Knowing the areas of the portions of the transverse sections that are cut off by the first water-line, and also the centres of gravity, we may readily find the moments about an axis in the first water-line. These moments are set down in the column headed ***Moments about Base-line.*** These moments are now treated by Simpson's multipliers, and the resultant functions of moments, set down in the last column for the calculation of the main appendage, are summed up, and the result (13.51) is divided by the sum of the multiples of areas (28.22), giving 0.47 for the distance of the centre of buoyancy of the main appendage below the first water-line. The distance below the load water-line is 8.47 feet.

We may now enter in the table on page 52 the displacements of the main body, the main appendage, and of other items, such as stem, keel, stern-post, rudder, propeller, and plating, together with the distances of the centres of buoyancy of these several items from the midship section.

The displacement of the items stem, keel, rudder, propeller, etc., and the positions of their centres of buoyancy are calculated from their known dimensions or from working drawings. Since the displacement of all these secondary items is not large compared with the displacement of the main body of the ship, extreme accuracy is not required.

The sum of the displacements of all of the several parts is equal to the displacement of the ship.

Multiplying the displacement of each item by its leverage, and summing up separately fore and aft, we get the moment of the displace-

ment about an axis in the midship section. This moment divided by the displacement gives the position of the centre of buoyancy forward of the midship section.

In the same table, below the main calculations, are entered the distances of the centres of buoyancy of the several items of the displacement of the ship, below the load water-line. Again multiplying the displacements by their leverages we get the moments of the several items about an axis in the load water-line; dividing the sum of these moments by the displacement we get the distance of the centre of buoyancy below the load water-line.

The part of the table headed *Metacentres* relates to the calculation of stability and will be explained later.

Modifications of Displacement Sheet.—In the first place it is evident that increasing the number of stations and water-lines has the effect only of increasing the number of columns and lines of the displacement sheet, but that there is no real additional complexity from this source.

To adapt the displacement sheet to the trapezoidal rule it is sufficient to omit the Simpson's multipliers and to note or to remember that the end ordinates are to be multiplied by one-half before entering them in the table. If we consider any column of ordinates which belong to a given water-line, its first and last ordinates at the top and bottom are to be multiplied by one-half; consequently all the ordinates in the first line belonging to the first station (at the bow, for example) are to be multiplied by one-half, except that the first and last ordinate of that line are to be taken half as large, proportionally, as other ordinates of the same line; consequently the ordinates at the corners of the table are to be multiplied by one-fourth before they are entered; again, all the ordinates for the first and last water-lines are to be multiplied by one-half, except the top and bottom ordinates, which are multiplied by one-fourth as already pointed out, since they come in the corners. This statement may seem a little confused, but will be clear if a displacement sheet is laid out. Of course the distances between stations and between water-lines are reserved, together with other factors, in a manner similar to that explained near the bottom of page 56. French naval architects consider this as equivalent to computing for a fictitious carene with

the stations and water-lines one unit (one foot or one meter) apart; afterwards the computation can be extended to the real ship by multiplying the results for the fictitious carene by the proper factors. When Tchebycheff's rule is applied to a displacement sheet all the ordinates are entered with their proper values. The stations and water-lines must be spaced at the intervals demanded by the rule (see page 5), so that in general a new set of lines must be drawn, and the labor of drawing such a set of lines detracts materially from the advantages of simplicity and brevity which the rule promises. When there is a set of lines with numerous stations it may be that the stations demanded by the rule will fall near those already drawn (or that most of them will); in such case the adjacent stations on the body plan may be used instead of the ones demanded by the rule, the water-lines can then be properly spaced as the rule requires, and the half-breadths measured approximately; this will be advantageous for a preliminary design which does not demand great accuracy; finally, there may be one or more pairs of stations which must be drawn and faired before the half-breadths are measured, if it should happen that they fall too far from the nearest stations of the original lines.

The displacement sheet on page 50 gives, as one of the results, the location of the centre of buoyancy of the carene. Suppose now that a certain set of lines should have nine water-lines; it is clear that a displacement sheet like that on page 50 could be made which would deal with the carene up to the third water-line above the keel, including all the stations (usually seventeen or more); and it is evident that such a displacement sheet would give the location of the centre of buoyancy of the carene up to the third water-line. Another displacement sheet could be made which should deal with the carene below the fifth water-line, and other sheets could deal with carenes below the seventh and below the ninth water-lines. Such a duplication of displacement sheets would call for repetition of much of the work which can be avoided by a proper combination of all into one sheet. A combined displacement sheet or extended displacement sheet may have the water-lines entered in groups of three, namely, 1, 2, and 3, then 3, 4, and 5, then 5, and 6, 7, and so on; each group of three water-lines may be followed by columns like those headed *Vertical Sec-*

tions" on page 51 for the computation of volumes and moments; these columns will be made up as on page 50 for the first group of water-lines, but for the second group will be made up by summations from the beginning; thus the first function for areas is 1.3 (equal to 0.0+0.0+1.3), and if the quantities on the same line in the third group should be 1.4, 1.6, and 1.7, the function for areas on the first line for that group will be

$$(0.0+0.0+1.3)+1.4+1.6+1.7=1.3+4.7,$$

and other summations may be made in like manner; the successive columns of vertical sections will be made in a similar way by summing up from the beginning with arrangements for taking advantage of earlier summations wherever possible. An extended displacement sheet will give the location of the centre of buoyancy for the several carenes treated and provide the means for drawing the curve of buoyancy on Fig. 28, and also, as will be seen later, for the curve of metacentres; this curve of metacentres must not be confused with the metacentric curve.

It is evident that the extension of the displacement sheet to provide for the curve of centres of buoyancy must have the water-line entered in groups of three when Simpson's rule is used; when the trapezoidal rule is used the water-lines can be entered in groups of two, or all the water-lines can be entered in the body of the table, which may extend up to the highest water-line, and then special summations can be made in the body of the table, taking half-ordinates where required, because such work with the trapezoidal rule is so much simpler than with Simpson's rule. Four or five points will locate the curve of buoyancy well enough, for it must, of course, pass through the zero-point of the curve of displacement. French naval architects take ten water-lines up to the load water-line and one above, making eleven in all; they compute the location of the centres of buoyancy for carenes below the sixth, ninth, tenth, and eleventh water-lines.

The displacement sheet used by the Bureau of Construction and Repair is based on the trapezoidal rule and provides for as many as twenty-four stations and fourteen water-lines, two or more of which may be above the normal load water-line; of course, fewer stations

and water-lines may be used at discretion. The computation for the location of the centre of buoyancy is made for each water-line above that at the top of the keel. The methods for this sheet are simple and readily understood with the sheet in hand, after the discussion already given, but it cannot be stated briefly and clearly without the sheet. It may be sufficient to say that the summation is made progressive from the beginning, and is checked by a summation over all, so that without undue labor there may be a very efficient check on the work together with the provision for calculating the locations of the centres of buoyancy already mentioned.

Leland's Displacement Sheet.—A very compact displacement sheet, devised by Mr. Walter S. Leland, is given after page 63. By its aid the displacement and the location of the centre of buoyancy and also the metacentric height may be calculated with sufficient accuracy in a comparatively brief time; the sheet also gives convenient forms for summation and efficient checks to insure correctness.

Simpson's rule is made the basis of this table, but with a radically different arrangement from that ordinarily used, as, for example, on page 50. There the ordinates or half-breadths are entered in the body of the table and each is multiplied by the appropriate Simpson's multiplier and the multiples are added. Here the ordinates which are to be multiplied by a given Simpson's multiplier are first assembled by themselves, are summed up and the sum is multiplied by the proper multiplier. Thus at the upper left-hand corner of the sheet the even stations are assembled and below them are the odd stations; the first and last of the odd stations will commonly have zero ordinates, as in the tables, but if the half-breadths have appreciable values at these stations, *half* of their values may be entered. The water-lines are assembled in three groups; first the even water-lines that have the multiple 4; then the odd water-lines 3 and 5, that are to be multiplied by 2; finally, the lowest and highest water-lines 1 and 7, that are to be multiplied by unity.

To compute the area of a section, as at the second station, add up the half-breadths at the water-lines 2, 4, and 6, thus:

$$5.7+10.3+12.7=28.7;$$

multiply by 4, giving 114.8. Do the same for the half-breadths at the water-lines 3 and 5, except that, of course, the multiplier is here 2. The half-breadths at water-lines 1 and 7 are added and the sum multiplied by unity, i.e., the sum is set down directly. Now add the products, giving

$$114.8+40.0+15.6=170.4.$$

This function for transverse area when multiplied by $\frac{1}{3}$ (coefficient for Simpson's rule) and the distance between water-lines will give the area of the transverse section at station 2. As in other displacement sheets, this area is not computed directly, but enters into the computation of the volume and displacement.

Taking now the second water-line, the half-breadths at the even stations are entered at the upper part of the table and summed up, giving 138.6, which is multiplied by 4 and the multiple is 554.4. In like manner the half-breadths at the odd stations are summed up and multiplied by 2, giving 277.2. The sum of these multiples, 831.6, is the function for the area of the second water-line, which when multiplied by $\frac{1}{3}$ and the distance between stations would give the area of that water-line; the areas of water-lines are not computed directly, but enter into the computation of the volume and displacement of the ship.

Returning to the column of functions of areas of transverse sections, they are, of course, grouped as the stations are, namely, the even stations at the top and the odd stations at the bottom. The functions of areas at even stations are added and the sum multiplied by the Simpson's multiplier 4, and the sum of functions of areas at odd stations is multiplied by 2; the sum of these multiples,

$$11136+5433.4=16569.4,$$

is the quantity which enters into the calculation of displacement under the head of *Results*. These results and the ***Principal Dimensions*** are assembled at the right of the sheet.

With this introduction and with a clear comprehension of the methods of the standard table on pages 50, 51, the reader should now have no difficulty in tracing through the computation of the displacement from the functions of water-lines, and the determination of the

LELAND'S DISPLACEMENT SHEET NO. 1.

Stations.	Levers.	Ordinates. Water-lines. 2	Ordinates. Water-lines. 4	Ordinates. Water-lines. 6	Sums and Multiples. 4	Ordinates. Water-lines. 3	Ordinates. Water-lines. 5	Sums and Multiples. 2	Ordinates. Water-lines. 1	Ordinates. Water-lines. 7	Sums and Multiples. 1	Functions of Areas.	Functions of Moments.	Metacentre. Cubes of Ordinates.	Metacentre. Ordinates × Lever.	Metacentre. Ordinates × (Lever)²	Appendage. ½ Area Sta. $+\frac{h}{3}$.	Appendage. ½ A. Sta. $+\frac{h}{3}\times$ Lever.
2	7	5.7	10.3	12.7	28.7 / 114.8	8.3	11.7	20.0 / 40.0	2.1	13.5	15.6	170.4	1192.8	246	94.5	661.5	1.2	8.4
4	5	19.5	22.5	23.3	65.3 / 261.2	21.5	25.0	44.5 / 89.0	15.3	23.4	38.7	388.9	1944.5	12813	117.0	585.0	26.8	134.3
6	3	23.3	24.0	24.0	71.3 / 285.2	23.6	24.0	47.6 / 95.2	21.6	24.0	45.6	426.0	1278.0	13824	72.0	216.0	46.1	138.3
8	1	23.4	24.0	24.0	71.4 / 285.6	24.0	24.0	48.0 / 96.0	22.6	24.0	46.6	428.2	428.2	13824	24.0	24.0	48.7	48.7
													4843.5 / 4558.3		307.5 / 350.0			329.4 / 389.3
10	1	23.4	24.0	24.0	71.4 / 285.6	24.0	24.0	48.0 / 96.0	22.6	24.0	46.6	428.2	428.2	13824	24.0	24.0	48.7	48.7
12	3	23.4	24.0	24.0	71.4 / 285.6	24.0	24.0	48.0 / 96.0	22.5	24.0	46.5	428.1	1284.3	13824	72.0	216.0	48.7	140.1
14	5	18.6	21.0	23.5	63.1 / 252.4	20.7	22.6	43.3 / 86.6	14.3	23.5	37.8	376.8	1884.0	12978	117.5	587.5	40.1	200.5
16	7	1.3	5.9	15.5	22.7 / 90.8	2.7	10.6	13.3 / 26.6	.5	19.5	20.0	137.4	961.8	7415	136.5	955.5	0.	0.
Sums........		138.6	155.7	171.0		148.8	163.9		121.5	175.9		2784.0	285.2	90962	−42.5	3269.5	260.3	−59.9
Sim. multiplier..		4	4	4		4	4		4	4		4	4	4	4	4	4	4
Multiple.......		554.4	622.8	684.0		595.2	655.6		486.0	703.6		11136.0	1140.8	363848	−170.0	13078.0	1041.2	−239.6

1	8	0	0	0		0	0		0	0				0	0	0	0	0
3	6	14.0	18.3	20.1	52.4 / 209.6	16.5	19.3	35.8 / 71.6	9.3	20.5	29.8	311.0	1866.0	8615	123	738	14.6	87.6
5	4	22.3	23.6	24.0	69.9 / 279.6	23.3	23.7	47.0 / 94.0	19.4	24.0	43.4	417.0	1668.0	13824	96	384	36.5	146.0
7	2	23.4	24.0	24.0	71.4 / 285.6	24.0	24.0	48.0 / 96.0	22.5	24.0	46.5	428.1	856.2	13824	48	96	48.7	97.4
9	0	23.4	24.0	24.0	71.4 / 285.6	24.0	24.0	48.0 / 96.0	22.6	24.0	46.6	428.2	4390.2 / 4247.0	13824	26: / 279	0	48.7	331.0 / 304.0
11	2	23.4	24.0	24.0	71.4 / 285.6	24.0	24.0	48.0 / 96.0	22.6	24.0	46.6	428.2	856.4	13824	48	96	48.7	97.4
13	4	22.5	23.5	23.8	69.8 / 279.2	23.4	23.6	47.0 / 94.0	20.1	24.0	44.1	417.3	1669.2	13824	96	384	42.5	170.0
15	6	9.6	17.0	21.6	48.2 / 192.8	13.7	19.6	33.3 / 66.6	5.0	22.5	27.5	286.9	1721.4	11391	135	810	6.1	36.6
17	8	0	0	0	0.	0	0		0	0				0	0	0	0	0
Sums		138.6	154.4	161.5		148.9	158.2		121.	1630		27167	143.2	89126	−12	250.8	245.8	27.0
Sim. multiplier		2	2	2		2	2		2	2		2	2	2	2	2	2	2
Multiple		277.2	308.8	323.0		297.8	316.4		243.0	326.0		5433.4	286.4	178252	−24	501.6	491.6	54.0
Sums of mults		831.6	931.6	1007.0		893.0	972.0		729.0	1029.6		16569.4	1427.2	542100	−194.0	18094.0	1532.8	−185.6
Sim. multiplier		4	4	4		2	2		1	1								
Func. of volumes		3326.4	3726.4	4028.0		1786.0	1944.0		729.0	1029.6		16569.4					1532.8	
Lever		5	3	1		4	2		6	0							6.33	
Func. of moms.		166320	11179.2	4028.0		7144.0	3888.0		4374.0	0			47245				9708.0	

RESULTS.

Displacement .... $(16569.4+1532.8)\frac{h}{3}\times\frac{s}{3}\times\frac{2}{35} = 7832.7$ tons

Area load water-line .... $1029.6\times\frac{25}{3} = 13865.3$ sq. ft.

Centre of gravity of load water-line .... $194\times\frac{s}{1029.6} = 4.0$ ft.

Transverse moment of inertia .... $542100\times\frac{s}{9}\times 2 = 243045$[illegible]

Transverse metacentric radius .... $2434450 \div 7832\times 35 = 8.89$ ft.

Moment of inertia about midship section .... $18094\times s^2\times\frac{s}{3}\times 2 = 993265$[illegible]

Area $\times 4^2 =$ .... $13865.3\times 4^2 = 2218$[illegible]

Longitudinal moment of inertia .... 9910468[illegible]

Longitudinal metacentric radius .... $99104685 \div 7832.7\times 35 = 361.5$ ft.

Centre of buoyancy forward of midship section .... $\frac{12416}{18102.2}\times S = 1.39$ ft.

Centre of buoyancy below load water-line .... $\frac{56953.2}{18102.2}\times h = 9.92$ ft.

Moment to trim one inch .... $\frac{7832.7\times 361.5}{35\times 323} = 730$ ft.-tons

Tons per inch of immersion .... $\frac{13865.3}{12\times 35} = 33$ tons.

PRINCIPAL DIMENSIONS.

Length over all	343 ft.
Length between perpendiculars	323 ft.
Beam moulded	48 ft.
Beam at load water-line	48 ft.
Depth	26 ft.
Depth, middle of side	26 ft.
Draught to load water water-line (No. 7)	21.16 ft.
Stations spaced, s	20.2 ft.
Water-lines spaced, h	3.16 ft.
Displacement of shell plating, propeller, and rudder	53.2 tons
Centre of buoyancy of same forward of midship section	1.39 ft.
Total displacement	7885.9 tons
Centre of buoyancy forward of midship section	1.39 ft.

$\times\frac{h}{3}\times\frac{s}{3}\times\frac{2}{35}$

$+620.8=262$

$\times 2=12025.$

$s\div 893=4.5$

$\times\frac{1}{3}\times\frac{s}{3}\times 2$

$7\div(2626.4\times$

$\times s^2\times\frac{s}{3}\times 2$

.3

$.3\div(2626.4\times$

$\frac{-185.6}{+1532.8}\times s=$

$\frac{+9708}{+1532.8}\times h=$

I. $=495$ tons

$=4.84$ tons

L. $=28.6$

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location of the centre of buoyancy and the calculation of the metacentric height. Perhaps attention should be called to the fact that it is intended that the first water-line shall be located so as to cut off a main appendage having transverse sections that are sensibly triangular, so that their areas and centres of figure are readily determined. The half-area of the appendage divided by $\frac{h}{3}$ ($\frac{1}{3}$ the distance between water-lines) gives a number for each station that may be combined with the functions for areas in the determinations of displacement and location of the centre of buoyancy. Thus the sum of multiples 1532.8 appears with the sum 16569.4 in the calculation of displacement in the results. This table has seven water-lines and seventeen stations; the method can evidently be applied to any number. Half-stations can be introduced with a little trouble if thought necessary.

A second form of this table, adapted to give points for the construction of curves of displacement up to various water-lines, of tons per inch of immersion, and of metacentric heights, is given at the end of the book. Here the ordinates at odd stations are separated from those for even stations, and, further, the columns of ordinates are set down in groups of three, such as 1, 2, 3, and 3, 4, 5, and 5, 6, 7, thus providing for computations for each alternate water-line of the several functions named. A peculiarity which should be carefully noted is that the half-breadths of water-lines are set down in the proper place, and that in a column following the group of three there is given at each station three times the middle ordinate of the group for that station; the sum of the three half ordinates in the group and the extra number is the function of area. For example, at the station No. 3 the ordinates and three times the middle ordinate give the sum

$$9.3+14.0+16.5+42=81.8,$$

which is equivalent to

$$9.3+4\times 14.0+16.5=81.8.$$

The first group of three columns is in fact a displacement sheet

for a carene with three water-lines giving the displacement, location of the centre of buoyancy, and metacentric radius. The second group of three columns is a displacement sheet giving the displacement between the third and fifth water-lines, which displacement added to that between the first and third water-lines gives the displacement of the carene below the fifth water-line. The calculation for location of centre of buoyancy and for metacentric radius is readily made for that carene below the fifth water-line. The third group carries the computation up to the load water-line. Most of the curves on Fig. 28, page 45, have a known zero-point and can be located fairly well by three additional points which can be determined by aid of the displacement sheet last described. If a better location of such curves is desired, more water-lines may be drawn in at regular intervals, or half-spaces for water-lines may be introduced. Four or five points in addition to the zero-point will give a sufficiently good location of the curves on Fig. 28.

CHAPTER III.

STABILITY.

Equilibrium of Floating Bodies. — As already stated in the chapter on displacement, the resultant of the pressure of the liquid on a floating body is a single force, equal to the weight of the displaced liquid and applied at its centre of figure, which point is known as the centre of buoyancy. The weight of the body is, of course, applied at the centre of gravity. If both the liquid and the floating body are at rest, the weight and the upward pressure (or buoyancy) of the liquid will be vertical. If the floating body, further, is affected by no other forces than its weight and the buoyancy of the liquid, it will be in equilibrium when they act along one line, that is, when the centre of buoyancy is on a vertical through the centre of gravity, as shown in Fig. 31.

If a ship is affected by a couple, its displacement will be unchanged, but it will take an inclination, as shown by Fig. 32, and the weight and buoyancy of the ship will form a couple equal and opposite to the inclining couple. The axis about which the ship is inclined is called the axis of inclination, and any plane perpendicular to the axis of inclination is called a plane of inclination. The axis of inclination for the figures is taken fore and aft and the plane of inclination is consequently transverse. The vertical through the new position B' of the centre of buoyancy is $B'Z'$, which makes an angle θ with the line GZ, the vertical before the ship was inclined; BZ is called the former or old vertical, while $B'Z'$ is called the new vertical. The moment of the weight and buoyancy is called the righting moment; it is equal to the product of the weight or displacement of the ship (in tons) by the arm GG' (in feet) perpendicular from G on to the vertical $B'Z'$. GG' is called the righting arm.

The new vertical $B'Z'$ intersects the old vertical in the point m. It is clear that when G is below m the couple of weight and buoyancy form a righting couple. Should the centre of gravity for any inclination be above m, this couple will be an upsetting couple; if a ship is inclined by any means to such a position that G is above m, it will if free capsize.

Metacentre.—For every inclination of a ship there is a corresponding position of the intersection of the new and the old verticals, as in Fig. 32. As θ, the angle of inclination, approaches zero, the inter-

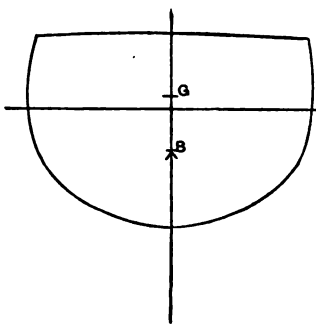


FIG. 31.

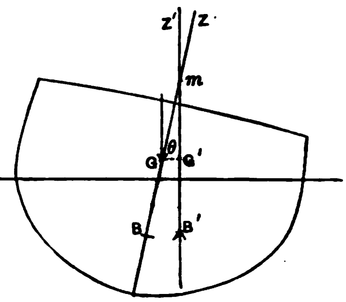


FIG. 32.

section of the old and new verticals approaches a limit which is called the metacentre. The metacentre is a definite, fixed point which depends on the form of the ship, the displacement, and the axis of inclination. It is customary to associate the term metacentre with the normal erect position of the ship; but for every position of equilibrium which a ship can be made to assume by shifting weights or by any means except the action of an external inclining couple, there will be a point which is properly the metacentre for that condition.

Metacentric Height.—When a ship is floating freely in equilibrium the distance from the centre of gravity to the metacentre is called the metacentric height. It is proper and it will be found convenient to associate with every position that a ship may take in consequence of a manner of loading or even in consequence of an accident,

such as admitting water to the hold, the corresponding metacentre and metacentric height. But it is customary to determine and record for each ship the metacentric height for the erect position and the normal displacement and trim. This is what is always meant by metacentric height unless otherwise specified. If a ship has a number of normal conditions, such as loaded and empty, the metacentric height for each of these conditions may be determined.

It has already been noted that curves of metacentres and curves of centres of gravity are drawn during the design of a ship, as shown on Fig. 30, page 49; from such a diagram the metacentric height for any condition may be taken at once.

Stable and Unstable Equilibrium.—If the centre of gravity and the centre of buoyancy of a ship are in the same vertical, they are in equilibrium; if the centre of gravity is below the metacentre, the equilibrium is stable and the ship, if disturbed slightly, will return to the position of equilibrium. If the centre of gravity is above the metacentre, the ship will leave the position of equilibrium and may capsize; many steamships are unstable when empty, but they usually are in no danger of capsizing, for they come to a position of equilibrium after a moderate inclination. Most ships have two positions of stable equilibrium, one erect in the normal position and the other when capsized. Self-righting life-boats are in unstable equilibrium when capsized, and immediately right themselves when released.

Statical Stability.—The moment of the couple formed by the weight and the buoyancy of a floating body, which is inclined from a position of stable equilibrium, is called the statical stability. The determination of the statical stability involves the calculation of the position of the centre of gravity and the position of the centre of buoyancy in the inclined position as well as in the initial position.

The centre of gravity of a ship may be determined by calculation from the weight, form, and position of all the members of the ship. For this purpose a record should be kept of the weight of all the members as they are worked into the ship; the location is commonly determined from the working drawings. An estimate may be made from the drawings and specifications of the scantlings of the ship,

of the location of the centre of gravity before construction is begun. In either case a very considerable weight of cement, paint, fittings, etc., eludes calculation and is estimated from the effect of such materials in ships already built. When the weights and positions of all the members of a ship are known, the centre of gravity is found by taking moments about a convenient axis. The principle of this calculation is simple, but the work is laborious and difficult. Certain devices for reducing the labor will be given in the chapter on strength.

The centre of gravity of a ship when afloat may be determined by making an inclining experiment; this is the converse of the calculation of stability, and will be dealt with after the discussion of stability.

The centre of buoyancy for the ship when initially inclined is determined by processes similar to those described in the previous chapter for the ship when erect.

Symmetrical Bodies.—If a floating body is symmetrical with regard to the plane of inclination through the centre of gravity, then the centre of buoyancy remains in that plane of inclination for all inclinations about the corresponding axis.

While large well-formed ships are not usually symmetrical fore and aft, the lack of symmetry of the under-water body for inclinations realized in service is not very marked, and it is customary to make the calculation of stability as though the centre of buoyancy remained in the plane of inclination, in the first place, and afterwards the amount that the centre of buoyancy deviates from that plane is found if desired. This deviation of the centre of buoyancy from the plane of inclination increases the stability; the increase is not large and is usually neglected, as that is on the safe side and saves labor.

The discussion of stability for transverse inclinations is much simpler for ships which are symmetrical fore and aft, and will serve as a good introduction to more complicated discussions. As all ships are symmetrical transversely, such a ship will be doubly symmetrical.

Couples of Form and Weight.—As has been shown, the statical stability of a ship that is symmetrical with regard to the plane of inclination is a couple in the plane of inclination. Thus in Fig. 33

B and B', the original centre of buoyancy and the new centre of buoyancy after inclination, are in the plane of inclination through G, the

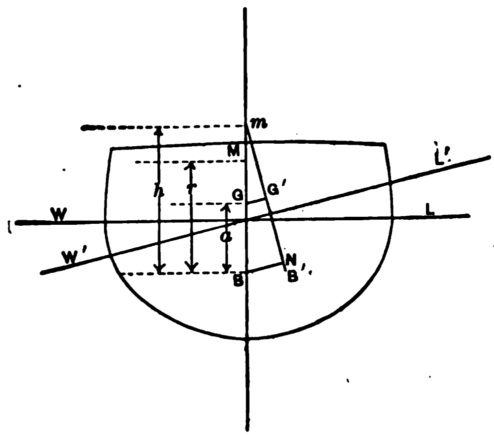


FIG. 33.

centre of gravity. In this figure, as is convenient and customary, the figure of the ship is erect and the water-line $W'L'$ is drawn at the angle of inclination θ with the original water-line WL. The righting couple, or statical stability, is

$$D \times GG', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which D is the displacement in tons, so that the units of moment are foot-tons. Now the location of B, the centre of buoyancy, for the erect position can be readily determined from the lines of the ship, while our knowledge of the location of the centre of gravity G is commonly only approximate, and, further, it is liable to change even for the same displacement, on account of the character or the stowage of the cargo, or other weights carried by the ship.

It is customary to represent the distance from the centre of buoyancy B to the intersection m of the old and new verticals by the letter h, and the distance from the centre of buoyancy to the metacentre M by r, while the distance of the centre of gravity above the centre of buoyancy is represented by a; all as indicated on Fig. 33. The metacentric height is clearly equal to $r - a$.

From the triangle mGG',

$$GG' = (h - a) \sin \theta;$$

consequently the statical stability is

$$D(h-a) \sin \theta = Dh \sin \theta - Da \sin \theta. \quad . \quad . \quad . \quad . \quad (2)$$

The expression $Dh \sin \theta$ depends only on the form of the ship, and is called the *righting moment of form;* the expression $Da \sin \theta$ depends on the position of the centre of gravity and is called the *righting moment of weight.*

Determination of Metacentric Radius.—For inclinations not exceeding 10° we may ordinarily substitute the distance of the metacentre above the centre of buoyancy MB, Fig. 33, for the distance Bm. The following simple demonstration of a method of calculating the metacentric radius will be found convenient at this place. A more complete discussion of the properties connected with the metacentre will be given later.

Let Fig. 35 represent the transverse section of a ship with the water-lines WL and $W'L'$ cutting off equal displacements, and let Fig. 34 be a perspective view of the wedge of immersion added by the inclination and the wedge of emersion that is taken away.

To find the location of the new position B' of the centre of buoyancy after an inclination, we may determine the moment of the new carene cut off by the water-line with reference to some convenient plane and divide that moment by the displacement; which displacement is equal to the original displacement cut off by the water-line WL. The most convenient plane for this purpose is a fore-and-aft plane through B perpendicular to the new water-line WL, because the moment of the original carene with reference to that plane is zero. Consequently the moment of the new carene can be found by adding the moment of the wedge of immersion and subtracting the moment of the wedge of emersion, both taken with reference to the plane through B.

The volumes of the two wedges are necessarily equal; and if the inclination is small, the water-lines WL and $W'L'$ can be assumed to intersect in a fore-and-aft line through O, the middle point of WL in Fig. 35.

If the angle of inclination $\Delta\theta$ is small, the area of a sector cut by a transverse plane from the wedge of immersion will be nearly

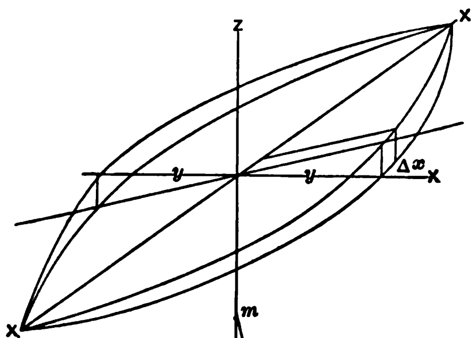
$$\tfrac{1}{2}y \cdot y\Delta\theta,$$

where y is the half-breadth of the original water-line; and the volume of a slice Δx long will be nearly

$$\tfrac{1}{2}y^2\Delta x\ \Delta\theta.$$

Let the distance OT, Fig. 35, be represented by s; then the

FIG. 34.



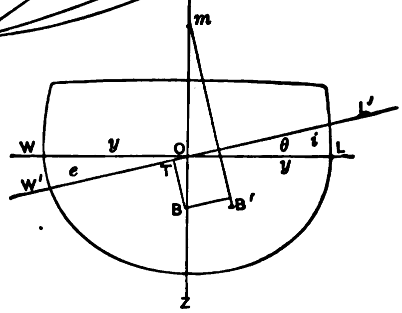


FIG. 35.

moment of a thin slice cut from the wedge of immersion will be nearly

$$\tfrac{1}{2}y^2\Delta x\ \Delta\theta(\tfrac{2}{3}y+s), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the moment of the corresponding slice cut from the wedge of emersion will be

$$\tfrac{1}{2}y^2\Delta x\ \Delta\theta(s-\tfrac{2}{3}y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a distance to the left being negative. Subtracting (4), the moment of the slice of the wedge of emersion, from that of the wedge of immersion (3), we have for the resultant moment

$$\tfrac{2}{3}y^3\Delta x\,\Delta\theta.$$

Integrating for the entire wedges, we have for the righting moment

$$\Delta\theta\tfrac{2}{3}\int y^3dx = i\Delta\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where i is the moment of inertia of the water-line about a fore-and-aft axis through its centre of figure. But the distance of the centre of buoyancy B' from the plane through B is (see Fig. 33)

$$h \sin \Delta\theta = r\Delta\theta,$$

since $\Delta\theta$ is small; and consequently the moment of the new carene with regard to the plane through B is

$$Vr\Delta\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where V is its volume in cubic feet. Equating this to the expression for the moment by equation (4),

$$Vr\Delta\theta = i\Delta\theta,$$

$$r = \frac{i}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here r is the transverse metacentric radius, i is the transverse moment of inertia of the water-line, and V is the displacement in cubic feet.

A corresponding expression for the longitudinal metacentric radius is

$$R = \frac{I}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Initial Stability.—The stability for a small inclination (ten degrees or less) is given approximately by the expression

$$D(r-a)\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where θ is the inclination in circular measure and is equal to the inclination in degrees multiplied by

$$\frac{\pi}{180}.$$

Metacentric Heights of the U. S. Light-ship.—The transverse moment of inertia of the load water-line of the U. S. Light Ship No. 51, on Fig. 26, measured by the integrator with the units in inches, is 1.36. Now the scale of the drawing is $\frac{1}{16}$ of an inch to one foot, or one inch is equivalent to 16 feet. Since the moments of inertia of similar figures vary as the fourth power of a linear dimension, the moment of inertia of the actual water-line, with the units in feet, is

$$1.36 \times 16^4 = 89200.$$

The volume of the displacement is 11957 cubic feet, consequently the transverse metacentric radius is

$$\frac{89200}{11957} = 7.5 \text{ feet.}$$

The longitudinal metacentric radius cannot be obtained by aid of the integrator, as it has not reach enough for that purpose.

Turning now to the displacement sheet for the light-ship, we find in the upper right-hand corner of page 51 a table headed *Metacentres*. In the first three columns we have the number of stations, the Simpson's multipliers, and the ordinates of the load water-line. These ordinates of the water-line take the place of y in the expression

$$I = \tfrac{2}{3}\int y^3 dx,$$

and the integration is replaced by the application of Simpson's rule to the cubes of the ordinates. Under the heading *Transverse* we have the cubes of the ordinates and the *functions of cubes* obtained by treating the cubes of ordinates with Simpson's multipliers. The sum of the functions of cubes is then multiplied by the constant

$$\frac{2}{3} \times \frac{13.43}{3},$$

the factor $\frac{2}{3}$ coming from the expression for the moment of inertia, and the factor $\frac{13.43}{3}$ being the coefficient for Simpson's rule. The moment of inertia, 90004.4, is then divided by the volume of the carene,

$$35 \times 341.65 = 11957.75 \text{ cubic feet,}$$

giving for the height of the transverse metacentre above the centre of buoyancy 7.53 feet.

The light-ship is not symmetrical fore and aft, and the centre of gravity of the load water-line is not in the midship section. The calculation for the position of the centre of gravity of the water-line is made as follows:

First, we have the calculation of the area of the water-line by Simpson's rule from the ordinates, in the column headed *Functions of Ordinates* under the heading *Transverse.* This column and its sum may be copied directly from the column for the third water-line in the main displacement table. The area of the load water-line is, therefore,

$$2 \times \frac{13.43}{3} \times 232.7 = 2083.44 \text{ sq. ft.}$$

To get the moment of the load water-line about an axis in the midship section, we may multiply each ordinate by its distance from that section and treat the quantities thus obtained by Simpson's rule; or it will be more convenient to multiply each ordinate by the number of spaces, leaving the distance between stations to be multiplied in afterwards. Under the heading *Longitudinal* we have the number of spaces of each ordinate from the midship section, with the title *Multipliers*, and following that column we have the products of the ordinates by the multipliers, under the title *Functions for Centre of Gravity of L. W. L.* The two halves of this column are summed up separately, and the difference of these sums, or the algebraic sum, is 25.2. The moment of the load water-line is consequently

$$2 \times \frac{13.43}{3} \times 13.43 \times 25.2,$$

in which the factor 2 is introduced to allow for both halves of the

water-line, the factor $\frac{13.43}{3}$ is the coefficient for Simpson's rule, and the factor 13.43 is the distance between stations reserved when the ordinates were multiplied by the number of stations. The centre of gravity of the load water-line is, moreover,

$$\frac{2\times\frac{13.43}{3}\times 13.43\times 25.2}{2\times\frac{13.43}{3}\times 232.7}=\frac{13.43\times 25.2}{232.7}=1.45 \text{ feet}$$

forward of the midship section.

To get the moment of inertia of the load water-line about an axis in the midship section, we may multiply each ordinate by the square of its distance from the midship section, and treat the quantities thus obtained by Simpson's rule; or, for convenience, we may multiply by the square of the number of spaces, reserving the square of the distance between stations. As we have already the products of the ordinates by the number of spaces (the functions for the centre of gravity), we may multiply these products once more by the number of spaces, thus getting the *Functions for the Moment of Inertia.* Since the ordinates in this operation are multiplied by the square of the number of spaces, the products are positive both forward and aft, and the whole column is summed up together. The moment of inertia of the load water-line about an axis in the midship section is, therefore,

$$2\times\frac{13.43}{3}\times(13.43)^2\times 827.2=1614.9\times 827.2=1335845.$$

But this moment of inertia is too large and must be reduced to an axis through the centre of gravity of the water-line. We therefore subtract from the quantity just obtained the product of the area of the load water-line by the square of the distance of the midship section from the centre of gravity of the water-line. Finally, the moment of inertia about a transverse axis through the centre of gravity is divided by the volume of the carene, giving for the distance of the longitudinal metacentre above the centre of buoyancy 111.4 feet.

The position of the centre of gravity for the light-ship is not known; it may be assumed to be at the load water-line, and such an assumption will probably not be much in error. Now the centre of buoyancy is 3.4 feet below the load water-line, consequently the assumption just made will give for the transverse metacentric height

$$7.5-3.4=4.1 \text{ feet,}$$

and for the longitudinal metacentric height

$$111.4-3.4=108 \text{ feet.}$$

For an inclination of 5° the stability of the light-ship calculated from the metacentric height will be

$$341.6\times3.4\times\frac{5\pi}{180}=101.5 \text{ foot-tons.}$$

It is evident that the calculation for metacentric heights may be made with equal or greater facility when the trapezoidal rule or Tchebycheff's rule is used.

Curve of Metacentres.—The method of computing the distance of the metacentre above the centre of buoyancy, or the metacentric radius, which has just been illustrated by an application to the U. S. light-ship, may evidently be made for any carene below any given water-line. A complete determination of the properties of a ship includes the determination of the metacentric radii for a series of water-lines corresponding to the location of a series of centres of buoyancy, and the construction of a curve of metacentres as shown on Fig 30. After the centres of buoyancy are located as described on page 48, the corresponding metacentre can be located on the same ordinates by laying off the metacentric radii upwards from the centre of buoyancy.

It is not customary to draw a curve of longitudinal metacentres, but the same information in a different form is given by a curve of moments to change trim.

Curve of Centres of Gravity.—The calculation of the centre of gravity of a ship from its construction and lading is easily stated but is in practice very laborious and likely to be uncertain. From

the dimensions and the density of the several members of the frame and plating and other parts of the ship the weights may be computed, and the summation of the several weights gives the weight of the hull. At the same time the centre of gravity of each member can be located and its moment about a base-line at the top of the keel can be found, and a summation of all the moments gives the moment of the hull. In like manner the weight and moment of the machinery may be found, and the weights and moments of other fixed loads like the armor and armament of a war-ship. The weight and moment of the cargo or other variable load may be determined in much the same way for any manner of loading. Finally, the total moment divided by the total weight gives the distance of the centre of gravity of the ship and cargo above the base-line at the top of the keel.

Another method of locating the centre of gravity by an inclining experiment is described on page 156.

After the centres of gravity for several conditions of the ship have been determined they may be located on ordinates through the corresponding centres of buoyancy, and a curve of centres of gravity may be drawn. This curve differs from the curve of centres of buoyancy and of metacentres in that it may have sudden and notable changes of direction due to adding (or taking away) large weights, whose centres of gravity are at a distance from the centre of gravity of the ship and cargo; the other curves mentioned depend entirely on the form of the ship, which (below the highest assumed water-line) is usually fair and continuous, so that the curves are usually fair and without rapid change in direction.

Moment to Change Trim One Inch.—The trim of a ship in the most general meaning may include the whole arrangement and condition, as of masts, sails, and loading. But the significance of the term here is the relation of the draught of the ship at the bow and stern. Large ships which may have to enter relatively shallow harbors are commonly designed to trim on an even keel; smaller ships, and especially yachts, may have greater draught at the stern or may have a strongly curved keel. In any case the normal trim is that at which the ship is designed to sail. The most ready way of determining the change of trim due to changing the location of a

weight, or of any other influence, is by aid of the moment to change trim one inch.

Suppose that a ship which is in normal trim has a weight weighing w tons moved toward the bow (or the stern) a distance of gg' feet; the inclining moment due to this change is

$$w \,.\, gg'. \qquad (1)$$

Then, if θ is the longitudinal inclination produced by this change, the longitudinal righting moment by an adaptation of equation (9), page 76, the equal righting moment is

$$D(R-a)\theta = w \,.\, gg'. \qquad (2)$$

If the length of the ship in feet is L, and if the change of trim is one inch, then

$$\theta = \frac{\frac{1}{12}}{L}, \qquad (3)$$

which substituted in (2) gives

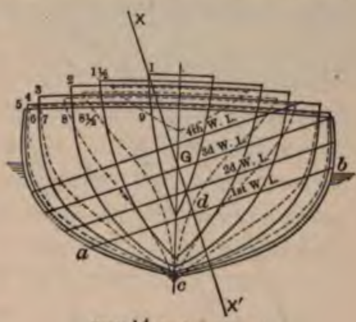
$$\text{Moment to change trim one inch} = w \,.\, gg' = \frac{1}{12}\frac{D(R-a)}{L}. \qquad (4)$$

Methods of Calculating Stability.—A large number of methods, or variety of methods, have been proposed for calculating the stability of a ship. An analysis of these methods, however, reduces them to two or three. The two methods in common use are known as the *method of cross-curves* and *Barnes' method*. When these methods are mastered all the varieties will be readily understood.

Cross-curves of Stability.—It is convenient for the application of this method to draw a double body plan as shown on Fig. 36, which gives the body plan of the U. S. Light-ship No. 51. The after-body plan is dotted, to more readily distinguish it from the fore-body, but this is not necessary in practice; sometimes the after-body is drawn in red for the same purpose. The body plan differs from that on Fig. 26 in that each section is completed by drawing in the deck, and that the line of the rail is omitted.

The mechanical integrator is now commonly used for determining the stability of a ship; its use not only makes a great reduction in the labor of calculation, but also renders the explanation of the method more simple.

On the double body plan a number of equidistant water-lines are drawn; there are usually three below the load water-line and one above, making five in all. The lowest water-line is drawn at a draught somewhat less than that which the ship has when launched. For simplicity, only four water-lines are used on Fig. 36.



Scale 1/16 inch = 1 foot.

FIG. 36.

With the aid of the integrator calculate the displacement of the ship in cubic feet and in tons up to each of the given water-lines; this gives the displacement of the ship at the several draughts and in the proper trim.

Assume that the centre of gravity of the ship is on the load water-line; a correction to allow for the real location of the centre of gravity can be readily made afterwards. This assumption is customary because the real location is commonly unknown when the design is at the stage in hand, and it is convenient and proper to make this convention whether the location of the centre of gravity is known or not, for the sake of uniformity and certainty in the methods and records of calculations for stability. This conventional location of the centre of gravity, as at G on Fig. 36, may be considered to give an origin or pole through which the axis for moments is to be taken.

Now draw an axis through the origin at G (the assumed centre of gravity), making a known angle with the original vertical; on Fig. 36 this axis is at an angle of 20°; other axes are drawn later at various angles, as 30°, 50°, and 70°. In practice it is customary to draw axes at every 10° or 15° up to or beyond 90°.

Draw a series of water-lines perpendicular to the new vertical; it is customary to have one water-line pass through the assumed centre of gravity or origin, as, for example, through G on Fig. 36, and

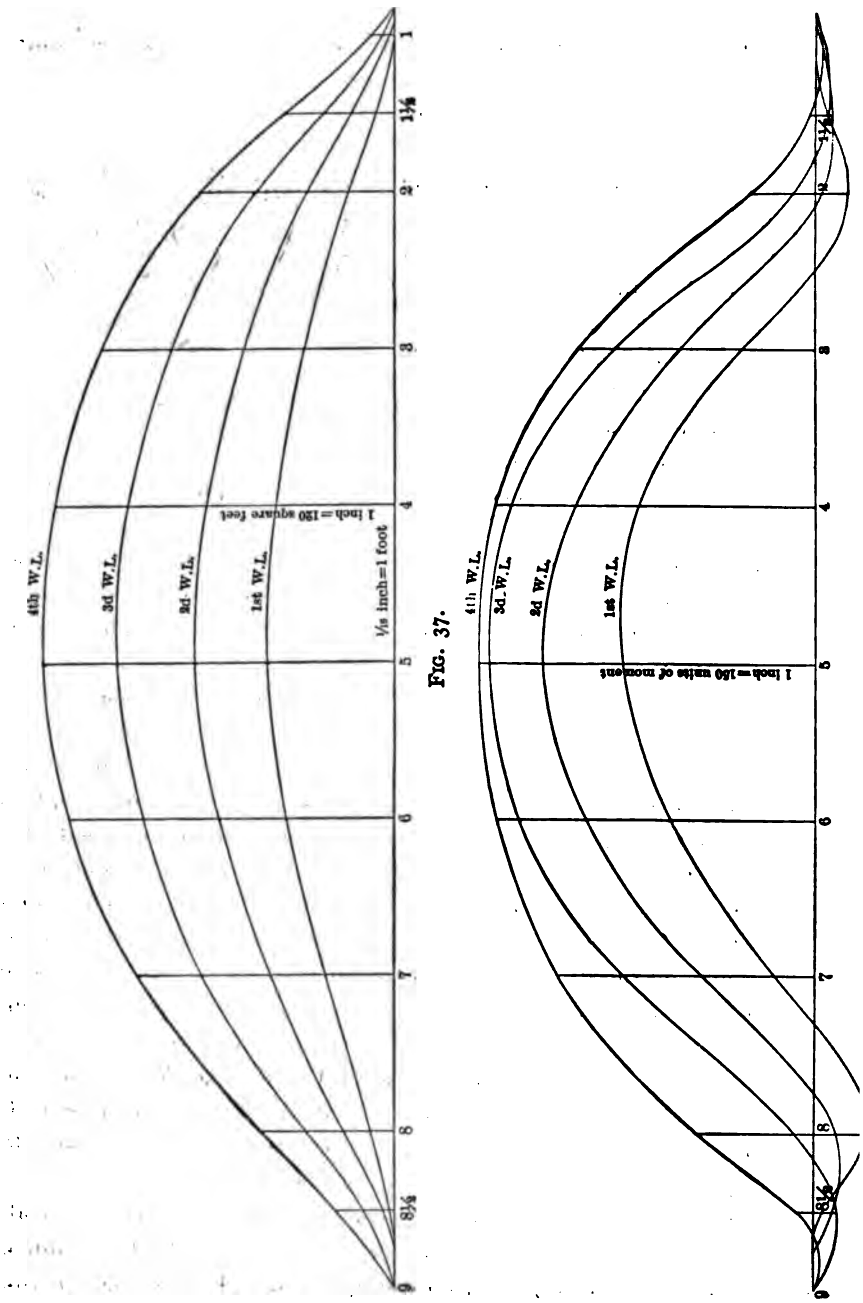

4th W.L.
3d W.L.
2d W.L.
1st W.L.
1/8 inch=1 foot
1 inch=120 square feet
1 inch=160 units of moments
1
1½
2
3
4
5
6
7
8
8½
9

FIG. 37.

it is also customary to space these lines at the intervals used for the original water-lines, but there is no necessity for following either custom.

Now adjust the integrator with its axis coincident with the inclined axis XX', and measure the area and the moment of each transverse section up to the first water-line. Plot these areas and moments as on Figs. 37 and 38, thus obtaining the *curve of areas of transverse sections and the curve of moments of transverse sections* for the first water-line and for 20° inclination. The area under the curve of areas is the volume of the carene up to the first water-line, and the area under the curve of moments is the moment of the same carene. The quotient of the moment by the volume gives the arm of the righting couple at the inclination of 20° for the volume of the carene below the first inclined water-line. The work with the integrator, and the calculations, are to be tabulated and arranged in a manner like that shown in Chapter II for the determination of the displacement by the aid of the integrator.

It is convenient now to plot the displacement for abscissa and the arm for ordinate on a diagram like Fig. 39, known as the cross-

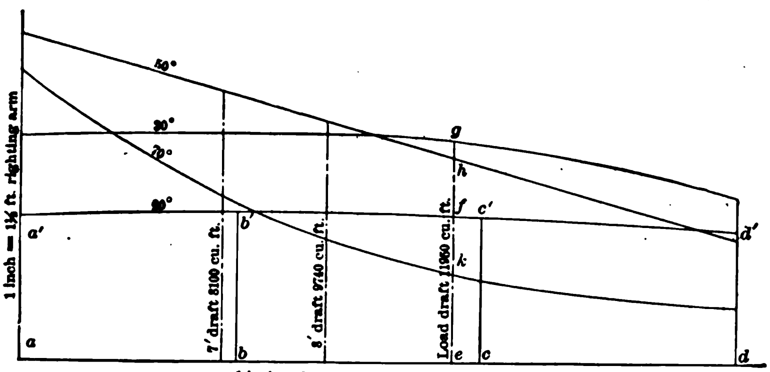


FIG. 39.

curves of stability; aa' is the ordinate for the first water-line and for an inclination of 20°. It so happens that this is the smallest displacement to be plotted on the diagram, and that the part of the diagram to the left of aa' can be omitted, thus saving space.

Now measure the areas and moments of the transverse sections

up to the second water-line and draw the curves of areas and moments for that water-line on Figs. 37 and 38, and find the arm of the righting moment, which is plotted on Fig. 39 at bb'. Repeating the process for the third and fourth water-lines, we get the curves of areas and curves of moments for those lines, as shown on Figs. 37 and 38, and from them we may get the arms of the righting moments for 20° inclination for the third and fourth water-lines, and may plot these arms at cc' and dd' on Fig. 39.

Through the points a', b', c', and d' a smooth curve is drawn which allows us to interpolate for the arm of the righting moment at 20° inclination, for any displacement. This curve is called the cross-curve for 20°.

Now draw a new axis at 30° from the original vertical through the point G, and carry through the work as for the 20° axis, thereby getting the cross-curve for 30° on Fig. 39. The cross-curves for 50° and 70° are obtained in a like manner.

If now we draw a vertical line on Fig. 39 at the abscissa representing a given displacement, for example the load displacement of 11950 cu. ft., the ordinates cut from this line by the several cross-curves will represent the arms of the righting moments for that displacement and at the given angles of inclination. Thus ef is the arm of the righting moment at 20° inclination, and eg, eh, and ek are the arms at 30°, 50°, and 70° respectively.

On Fig. 40 the angles of inclination are laid off as abscissæ, and the corresponding arms of the righting moments are laid off as ordinates giving the points f, g, h, and k, through which points and the zero-point a smooth curve is drawn, from which we may find the arm of the righting moment for any angle of inclination when the ship has the load draught of 9 ft. erect and the volume of carene of 11950 cu. ft.

Vertical lines drawn on Fig. 39 at the abscissæ representing the volumes 9740 cu. ft and 8100 cu. ft. at the draughts of 8 ft. and 7 ft. give the ordinates for the curves corresponding to those draughts on Fig. 40.

The curves on Fig. 40, showing the arms of the righting moments at all angles of inclination, are known as *curves of statical stability*. The statical stability for any condition is, of course, obtained by

multiplying the arm of the righting moment by the displacement in tons, and the corresponding moment is stated in foot-tons. Thus the arm of the righting moment for the normal displacement and for an inclination of 20° is 1.5 ft., and the statical stability is

$$342 \times 1.5 = 512 \text{ foot-tons,}$$

provided that the centre of gravity is on the load water-line.

The curves of statical stability on Fig. 40 are drawn with angles in degrees for abscissæ and with righting arms for ordinates. For certain purposes, as for the determination of dynamical stability (to

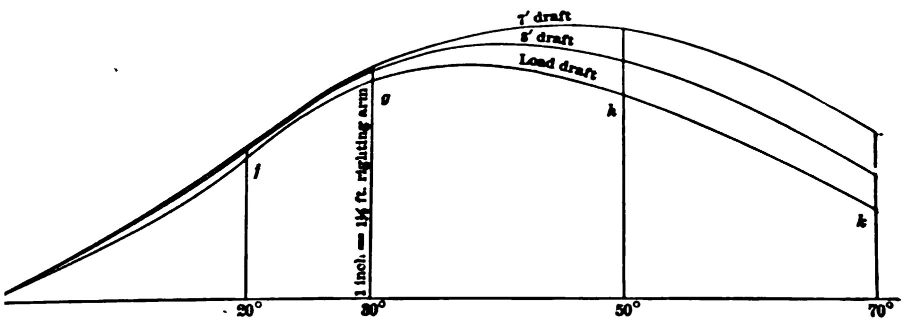


FIG. 40.

be discussed in the next chapter), it is necessary to have the abscissæ represent the angles in circular measure ($180° = \pi$) and to have the ordinates represent the righting moments in foot-tons. By changing the scales in the proper manner a diagram which is drawn with degrees and righting arms may be transformed to correspond to these requirements, or by the application of proper factors computations may be based on the diagram as drawn. It may be found convenient to draw the diagram with abscissæ representing angles in circular measure and ordinates in foot-tons for a particular purpose.

Correction for Change of Centre of Gravity.—Suppose that the centre of gravity of a ship is raised, without changing any other property, from the middle of the load water-line at G, Fig. 41, to some other position, as G' vertically over G. It is evident that

the righting arm is now reduced by the amount

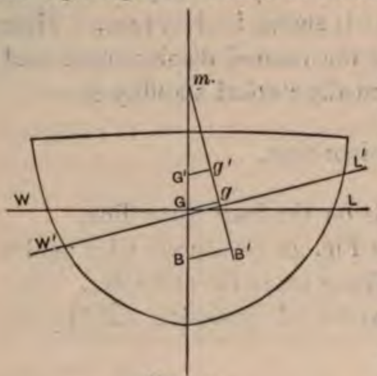


FIG. 41.

$$Gg - G'g' = GG' \sin\theta. \qquad (1)$$

Should the centre of gravity be depressed, the righting arm will be increased.

To transform the curve of stability, allowing for the true position of the centre of gravity, compute the change in righting arm by the method shown by equation (1) for each angle, subtract (or add) the correction from (or to) each ordinate, and draw a new curve like the curves of Fig. 40, except that the arms are shorter (or longer if G' is below G). It is evident that the corrections may be computed in anticipation and added to (or subtracted from) the arms derived from the cross-curves, and that the true curves of stability with the centre of gravity in its proper place may be drawn at once, without drawing curves for the pole at the middle of the load water-line; in practice this latter method is preferable, as it gives less chance for confusion in the record.

Abbreviations.—In order to exhibit the principles and methods of determining stability, the graphical process has been fully developed, including the construction of curves of areas and moments. For preliminary design it is important that the determination of stability shall be as brief and simple as possible, even at the expense of accuracy, so that the designer may estimate the nautical properties of the ship in question at an early stage of the design. In fact, extra refinement is unnecessary for any stability work, which is really used only as a basis for estimating the probable nautical qualities of a ship from comparison with ships already built and tried. It is proper, therefore, to take only a few stations and to use either the trapezoidal rule or Tchebycheff's rule. When the trapezoidal rule is used seventeen stations will be found sufficient in all cases, and of these the end stations may be neglected as their areas and moments are small. With Tchebycheff's rule nine stations are sufficient; in this case the end stations do not come at the ends of the ship. A combination of the use of the integrator for measuring the areas and

moments of the transverse sections and one of these rules for computing therefrom the volume and moment of the careen is very advantageous for determining stability. When the integrator is set to the required axis, as XX', Fig. 36, page 83, we may begin at any point, as for example at d on the lowest water-line, and may trace without stopping alı sections up to this line, omitting the sections at the bow and the stern; the difference of the readings of the area and moment wheels of the integrator at the end and at the beginning of this operation when multiplied by the proper factors for the integrator and the scale of the drawing give the sum of the areas of all the stations and the sum of the moments of all these stations about the axis XX'. These sums multiplied by the common interval between stations in the usual way of summing up by the trapezoidal rule will give at once the volume and the moment of the carene; and the moment divided by the volume will give the righting arm.

As an example of this method we have for the initial and final readings of the integrator and the differences of readings when all the stations (except the first and the last) on Fig. 36 are traced up to the first water-line, beginning and ending at the point d:

Areas..................0.942 − 0.586 = 0.356
Moments...............0.1145 − 0.1017 = 0.0128

The scale of the drawings is $\frac{1}{16}$ of an inch to a foot, and the factors for areas and moments are 20 and 40; consequently the differences given above are to be multiplied by

$$20 \times 16^2 \text{ for areas,}$$
$$40 \times 16^3 \text{ for moments,}$$

and the righting arm is equal to

$$\frac{40 \times 16^3 \times 0.0128}{20 \times 16 \times 0.356} = \frac{2 \times 16 \times 0.0128}{0.356} = 1.15$$

of a foot instead of 1.18 of a foot as measured on Fig. 39, ordinate aa'. Here, as on page 33, a greater accuracy of reading than is possible is attributed to the integrator, the actual results having been obtained from a body plan on a larger scale and then reduced pro-

portionally to get the illustrative computation to agree with the reduced figures given in the cut.

From the above computation it appears that with the integrator commonly used for ship-work, which has the factors 20 and 40 for areas and moments, the instrumental result (difference of readings) for moments is to be multiplied by 2 and by the scale of the drawing, and then divided by the instrumental results for areas, to get the righting arm. The volume of the carene (represented by the denominator of the left-hand quotient above) is to be divided by 35 (or by 36 for fresh water) to find the corresponding displacement in tons which is required for the abscissa of the cross-curve.

In like manner the displacement and righting arms for the carene up to the second water-line can be obtained after tracing all the stations (except the end stations) up to that line; and so on for the other water-lines; and again the same work may be done for all the other inclinations until the cross-curves are completed.

It will be found advantageous, after tracing the stations up to the first water-line, to trace only the additional areas between the first and second water-lines, and to get the instrumental results for the second water-lines by adding the quantities for areas and moments thus obtained to those previously obtained for the first water-line. For the figures selected are smaller and simpler and more easily traced and there is less chance of confusion in reading the recording-wheels for areas and moments.

Direction of Cross-curves.—The following method of determining the direction of cross-curves of stability, due to Naval Constructor

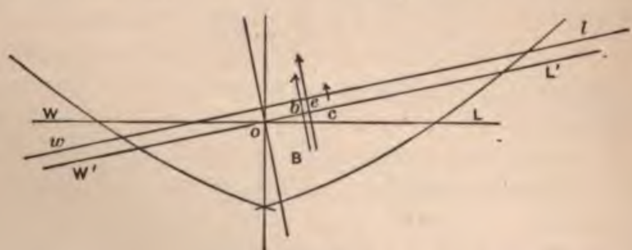


FIG. 42.

D. H. Taylor, U.S.N., makes it possible to definitely locate such curves with only a few points and proportionally reduce the labor

of computation. In Fig. 43 let RR' be a part of the cross-curves of stability for the ship shown by Fig. 42, the coordinates of the point R on that curve being the displacement and the righting arm for the carene below the inclined water-line $W'L'$. If the displacement is increased by the addition of a thin layer bounded by the water-line

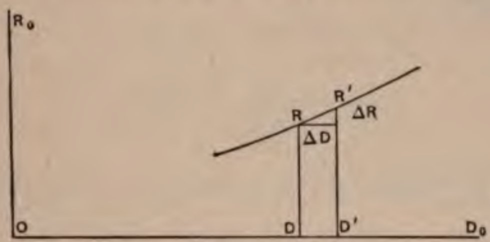


FIG. 43.

wl, the righting arm will, at the same time, receive an increment; in Fig. 43 the increments to the displacement and to the righting arm are represented by ΔD and ΔR; the tangent of the angle between the chord RR' and the axis OD_0 is

$$\tan\alpha = \frac{\Delta R}{\Delta D},$$

and at the limit when the thickness of the added layer approaches zero the tangent of the angle between the tangent to the curve at R and the axis OD_0 is

$$\tan\alpha = \frac{dR}{dD}.$$

To find an expression for this differential coefficient we may consider that the buoyancy of the carene below the water-line $W'L'$, Fig. 42, acts vertically through B along Bb; as the added layer is thin, its buoyancy may be considered to act at c, the centre of gravity of the water-line $W'L'$. Both Ob and Oc are readily found; the first is the righting arm, and the second is equal to the quotient of the moment of the water-line $W'L'$ about a longitudinal axis through O by the area of that water-line; the area and moment of the water-line can be computed from the breadths of the water-line at the several stations by the trapezoidal rule or by Tchebycheff's rule.

The determination of the buoyancy and its line of action follows

the same rules as the determination of weight and centre of gravity; consequently the line of action of the buoyancy of both the carene and the added layer divides the distance bc inversely proportional to the displacements D of the carene and ΔD of the added layer; but the addition of the layer increases the righting arm by the amount ΔR, consequently

$$D : \Delta D :: bc - \Delta R : \Delta R,$$

$$\therefore \tan \alpha = \frac{\Delta R}{\Delta D} = \frac{bc - \Delta R}{D},$$

and when the thickness of the layer approaches zero this becomes

$$\tan \alpha = \frac{dR}{dD} = \frac{bc}{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

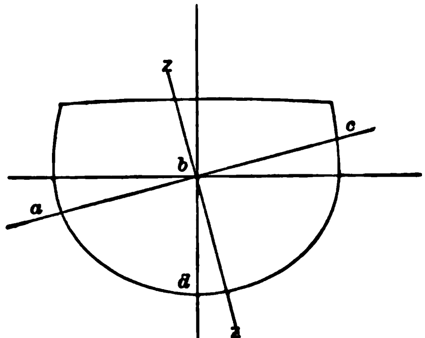
In this equation the distance bc is in feet and the displacement is in tons. On the diagram Fig. 43 the tangent of the angle depends also on the scale of the drawing. If the abscissæ are x tons per inch, and the ordinates are y feet per inch, then the value of $\frac{dR}{dD}$, computed by equation (1), is to be multiplied by $\frac{x}{y}$ before it is used as the tangent of the angle of the cross-curves in Fig. 43.

Single Body Plans.—The body plan of a ship is usually drawn single, as on Fig. 26, when the right hand represents the bow and the left hand the stern; and it is sometimes convenient to be able to use such a body plan for stability calculations; this can be done by drawing axes on both sides of the original vertical, as at ZZ and $Z'Z'$, Fig. 45. For the moment of the area of $abcd$ about the axis ZZ, Fig. 44, is equal to the sum of the moment of cbd about ZZ and of the moment of $a'bd$ about $Z'Z'$. To offset the advantage of using a body plan that is already drawn there is the labor of taking a double set of instrumental readings with the integrator set first to one axis and then to the other, and there is also danger of confusion. Measurements for numerical calculations of stability can be readily taken from a double body plan and without much chance of confusion.

Mechanical Devices.—If for any reason it is not desirable to draw axes and water-lines on a body plan that may have been prepared

for another purpose, we may, of course, protect the original body plan by a piece of tracing-cloth laid over it, on which lines may be drawn and the integrator may be run as required. If we choose, we may draw only one axis and the accompanying water-lines on the super-

FIG. 44.



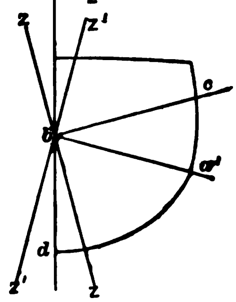


FIG. 45.

posed tracing-cloth and then may shift it to the successive angles of inclination required for the work of determining stability.

Or by a simple inversion of the above operation we may shift the body plan instead of the axis and water-lines. For this purpose a body plan should be drawn on a circular piece of tracing-cloth and slipped under the fixed tracing-cloth which bears the axis and water-lines; the middle point of the load water-line is to be placed under the origin or pole through which the axis of the integrator passes, and the body plan is set at the proper inclinations by aid of radial lines drawn through the middle of the load water-line for that purpose. After the integrator track is once adjusted it is not disturbed. To shift the body plan, thrust a needle through both pieces of tracing-

cloth at the pole, then lift one side of the upper tracing-cloth, and taking hold of the edge of the body plan, turn it through the desired angle. With some care and delicacy of handling, this method, though apparently crude, may be made to serve very well, and any error in placing the body plan is at once evident, so that unknown errors will not creep in.

At the shipyard of Messrs. Denny at Dumbarton a carefully made turntable on a vertical axis and with its edge divided into degrees is arranged for stability work; the pole or origin of the body plan is set over the axis of this table, and the table and body plan may be readily and accurately set to any angle. The integrator has its track on a fixed table adjacent, set so that the axis of the integrator shall pass through the pole; in order that the integrator shall not run off the fixed table on to the turntable, the arm is lengthened; this insures that the integrator wheels shall run on a proper surface on the fixed table, which surface can be kept in good condition.*

Numerical Calculation.—If an integrator is not at hand, the method of cross-curves may be carried on numerically, arranging the work in a form similar to the displacement table.

An axis is drawn at the chosen inclinations in Fig. 36, and inclined water-lines are drawn as on that diagram; additional lines are also drawn across the sections at the same intervals as the water-lines already described for the diagram. The width of each section is measured from the axis on the water-line bounding the section at the top, and on all the lines parallel to it, which cross the section. As the sections are not symmetrical with regard to the axis, the two sides must be recorded and calculated separately. It is customary to ignore the remnant of any section below the lowest parallel line crossing it, and to use the trapezoidal rule for calculations.

The widths of the sections measured from the axis, treated by the trapezoidal rule, give the areas of the sections, and these areas, treated again by the trapezoidal rule, give the volume of the inclined carene. The half-squares of the widths treated by the trapezoidal rule give the moments of the sections about a longitudinal axis, and these moments, treated again by the trapezoidal rule, give the moment

* Inst. Naval Archts., Vol. XXVIII.

of the carene, and this divided by the volume gives the arm of the righting moment.

The usual custom is to take the pole at the middle of the load water-line as in the work already described. Sometimes the pole is taken at the top of the keel. In either case allowance is made for the real location of the centre of gravity of the ship.

Leland's Method.—Mr. W. S. Leland has succeeded in getting satisfactory cross-curves by the following method, which appears to require the minimum labor. In the design of the ship he fairs the lines with seventeen stations and five water-lines. Taking Tchebycheff's rule with nine ordinates he finds that they come so near the second, fourth, fifth, eighth, ninth, tenth, thirteenth, fourteenth, and sixteenth stations of the seventeen already drawn that these stations may be taken instead of the sections demanded by the rule. He uses only three water-lines for stability calculations, one of which is above the load water-line; this gives only three points on each cross-curve, but he determines also the direction of the curves at the points selected by a method to be described later, and thus locates them satisfactorily.

In the determination of the righting arms and the displacements of the carene below the lowest water-line used for computing stability, he measures the areas and moments up to that line at the stations near those demanded by Tchebycheff's rule, and then applies that rule for the determinations of volumes and moments on which the computations of arms and displacements are based; this is done for a sufficient number of inclinations and locates one point on a cross-curve for each inclination. The remainder of the work is more expeditiously carried on numerically, for the part of any section or station between two of the adjacent water-lines (such as the first and second lines on Fig. 36) is very nearly a trapezoid, and consequently the additional areas and moments required to work up to a higher water-line are readily computed by the trapezoidal rule. The summations longitudinally for the arm and displacement are made as before by Tchebycheff's rule. The following form will be found convenient for his process.

In the space headed Integrator Work are given the readings for area and moment of the Amsler integrator after tracing in succession

LELAND'S METHOD FOR STABILITY.

Stations.	Technical Ordinates.	Integrator Work to Water-line 1.	30° Inclination.											
			Water-line 1.				Water-line 2.				Water-line 3.			
			r_1	r_1^2	r_2	r_2^2	r_1	r_1^2	r_2	r_2^2	r_1	r_1^2	r_2	r_2^2
2	1		7.5	56	11	121	14	196	12	144	18	324	12	144
4	2	Area = 2.033	20.3	406	19	361	15	625	23	529	19.5	380	23.5	552
5	3		22	484	20.5	420	25.5	650	25.5	650	17	289	25	625
8	4	Mom. = .412	22	484	20.5	420	25.5	650	26	676	9.5	90	26	676
9	5		22	484	20.5	420	25.5	650	27.5	756	9.5	90	26	676
10	6		22	484	19.7	388	25.5	650	27.5	756	9.5	90	26	676
13	7		22	484	20	400	18.5	342	27.5	756	9.	81	25	625
14	8		21.5	462	18.5	342	19.	361	22	484	9	81	23	529
16	9		0	0	7.5	56	22	484	8.4	71	12	144	11	121
Sum = Σ			159.3	3344	157.2	2928	200.5	4608	199.4	4822	113.0	1569	197.5	4624
Func. Area = $\Sigma(r_1+r_2) = fA$			$159.3+157.2=316.5$				$200.5+199.4=399.9$				$113+197.5=310.5$			
Func. Mom. = $\frac{1}{2}\Sigma(r_1^2-r_2^2) = fM$			$\frac{1}{2}(3344-2928)=208$				$\frac{1}{2}(4608-4822)=-107$				$\frac{1}{2}(1569.-4624)=-1528$			
C.G. of W.L. = $fM \div fA = Oc$			$\frac{208}{316.5}=.657$				$\frac{-107}{399.9}=-.268$				$\frac{-1528}{310.5}=-4.92$			
Disp. Layer = $(fA_1+fA_2)\times\frac{1}{2\times35}\times\frac{L}{9}\times h=$			0				$\frac{316.5+399.9}{2\times35}\times6\times\frac{323}{9}=2205$				$\frac{3999+310.5}{2\times35}\times6\times\frac{323}{9}=2187$			
Total Disp. = $\varDelta$			$2.033\times20\times\frac{323}{9}\times64\times\frac{1}{35}=2668$				$2668+2205=4873$				$4873+2187=7060$			
Mo. Layer = $(fM_1+fM_2)\times\frac{1}{2\times35}\times\frac{L}{9}\times h=$			0				$\frac{208-107}{2\times35}\times6\times\frac{323}{9}=311$				$\frac{-107-1528}{2\times35}\times6\times\frac{323}{9}=-5040$			
Total Mom. = M			$.412\times40\times\frac{323}{9}\times512\times\frac{1}{35}=8632$				$8632+311=8943$				$8943-5040=3903$			
Right Arm = $M \div \varDelta = R$			$8632\div2668=3.23$				$8943\div4873=1.83$				$3903\div7060=.554$			
Tangent = $\frac{Oc-R}{\varDelta}\times$ scale factor =			$\frac{.657-3.23}{2668}\times300=-.288$				$\frac{-.268-1.83}{4873}\times300=-.129$				$\frac{-4.92-.554}{7060}\times300=-.232$			
Angle =			$-16°.10'$				$-7°.20'$				$-13°.10'$			

all the transverse sections up to the third water-line; the reading for area multiplied by the integrator constant 20 and by the factor for the scale of the drawings gives the volume of the carene up to the third water-line; in like manner the reading for moment multiplied by the integrator constant 40 and by the factor for the scale of the drawing gives the moment of the carene. These will be referred to again.

The half-breadths of the inclined carene, which here are unequal, are measured at each of the first, second, and third water-lines for each station, and are entered as r_1, r_2, the subscript 1 being for the immersed side. The squares of the half-breadths are also entered for the computation of moments. The functions of areas of water-lines are obtained by summing the columns for r_1 and r_2 and adding the results; the functions for moments are obtained by summing the columns for r_1^2 and r_2^2 and taking the difference; thus for the first water-line we have

$$159.3 + 156.2 = 316.5, \text{ function for area;}$$
$$\tfrac{1}{2}(3344 - 2928) = 208, \text{ function for moment.}$$

The centre of figure or c. g. of the water-line is at a distance

$$316.5 \div 208 = 0.657 \text{ foot}$$

from the plane to which moments are referred. This quantity appears in the computation of the angle of the cross-curve.

If the length of the ship is 323 feet and if the distance between water-lines is 6 feet, then by Tchebycheff's rule the displacement of the layer between the water-lines 1 and 2 is

$$\frac{316.5 + 399.9}{2 \times 35} \times 6 \times \frac{323}{9} = 2205 \text{ tons,}$$

which is set down in the column for water-line 2, where it belongs; the corresponding quantity for water-line 1 is zero.

The total displacement Δ up to the third water-line is 2668 tons, found from the integrator work; the total displacement up to the

second water-line is this quantity plus the displacement 2205 tons of the layer between the first and second water-lines.

The moment of the layer between the first and second water-lines is, by Tchebycheff's rule,

$$\frac{208-107}{2\times 35}\times 6\times\frac{323}{9}=311 \text{ foot-tons,}$$

which is set down under the column for water-line 4.

The total moment up to the first water-line from the integrator work is

$$0.412\times 40\times\frac{323}{9}\times 512\times\frac{1}{35}=8632 \text{ foot-tons.}$$

Here 0.412 is the integrator reading, 40 is the integrator factor, and 512 is the factor for the scale of the drawing. The total moment for the carene below the second water-line is this quantity plus the moment for the layer between the first and second water-lines. The righting arm is of course the total moment divided by the total displacement for the carene in question.

The computations conclude with the determination of the direction of the cross-curve by the method on page 91. By this method the tangent to a cross-curve at a certain point makes an angle with the horizontal axis determined by the expression

$$\tan\alpha=\frac{bc}{D},$$

where bc, as in Fig. 42, is the distance of the centre of gravity of the water-line from the new vertical through the centre of buoyancy. In the table the distance of the centre of gravity of the first water-line from the pole O is $Oc=0.657$ of a foot, and the righting arm is

$$R=Ob=3.23 \text{ feet,}$$

so that the angle is determined by

$$\tan\alpha=\frac{0.657-3.23}{2668}\times 300=-0.288,$$

$$\alpha=-16^\circ\ 10'.$$

Here 2668 is the displacement up to the first water-line, and 300 is the factor for the scale of the cross-curves.

The table given on page 96 is for one inclination only, but the same arrangement can evidently be made for as many inclinations as desired, thus obtaining the cross-curves from which stability curves may be determined.

Barnes' Method.—This method of determining stability, which is called after the English naval architect who devised it, proceeds directly to determine the righting arm for any inclination without the intervention of auxiliary diagrams like cross-curves. Let Fig. 46 represent the midship section of a ship that is symmetrical fore and aft as well as transversely; let G be the centre of gravity, and B the centre of buoyancy in the erect position. When the ship is inclined to the angle θ the water-line is $W'L'$ and the corresponding centre of buoyancy is B', through which is drawn the vertical line $B'm$, intersecting the original vertical at m. The righting couple has the moment

$$D(h-a)\sin\theta = Dh\sin\theta - Da\sin\theta,$$

in which D is the displacement in tons, and h and a are distances from the centre of buoyancy to the intersection m, and to the centre of gravity G. It is convenient to separate the righting couple of form $Dh\sin\theta$ for calculation, and afterwards subtract the righting couple of weight $Da\sin\theta$.

Now take moments with reference to a fore-and-aft plane through the original centre of buoyancy B and parallel to the new vertical $B'm$. The moment of the carene bounded by the original water-line WL is, of course, zero. The moment of the carene bounded by the water-line $W'L'$ can be obtained by adding the moment of the immersed wedge and subtracting the moment of the emersed wedge; the latter is negative with regard to the axis through B, and when subtracted algebraically becomes positive.

Since the weights of the wedges are equal, the difference of their moments is the same whatever may be the axis about which moments are calculated. We may therefore choose an axis through e (Fig. 46), the intersections of the water-lines WL and $W'L'$. This will be a convenience in any case, but more particularly if the calculation is

made by Simpson's rule (or the trapezoidal rule) for angular areas. If an integrator is used, it is adjusted to an axis through e parallel to $B'm$, and the areas and moments of the sections of the wedge of immersion are determined in the manner explained in the chapter on displacement and centre of buoyancy. From the areas and moments of the sections the volumes and moments of the wedges may be calculated by the trapezoidal or by Simpson's rule; or curves of areas and moments may be plotted whose areas will give the desired volume and moment.

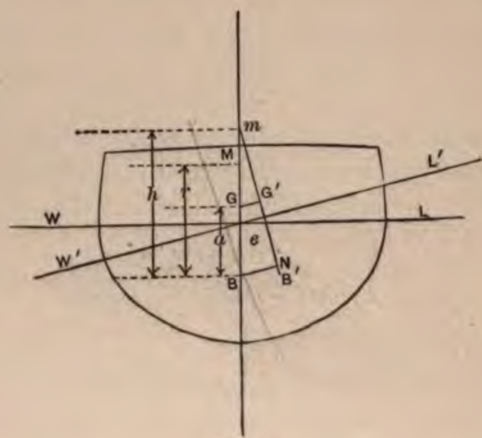


FIG. 46.

The emerged wedge is treated in the same way. Finally, the difference of the moments of the wedges divided by the displacement of the ship in cubic feet gives the arm BN (Fig. 46) of the righting couple due to form. The centre of buoyancy moves from B to B' as the ship is inclined from WL to $W'L'$, and BN is the component of its motion parallel to $W'L'$. Thus far the water-line $W'L'$ is assumed to be so drawn that the carene cut off by $W'L'$ shall be equal to the original carene; such a location of the inclined water-line can, in general, be obtained only by a process of trial and error, which is laborious if exact results are required, more especially if calculations are made by Simpson's rule without an integrator. If an integrator is at hand, and if the volumes and moments of the wedges may be calculated with sufficient accuracy by applying the trape-

zoidal rule to the areas and moments of the sections of the wedges, then we may apply the rapid method described on page 39, running the integrator in succession around all the transverse sections of a wedge and taking readings at the beginning and end only. The volumes of the wedges will, of course, be equal, if the corresponding integrator readings for areas give the same results for the two wedges. Such a method is sometimes followed in yacht-work, to which Barnes' method is well adapted to determine the power of the yacht to carry sail.

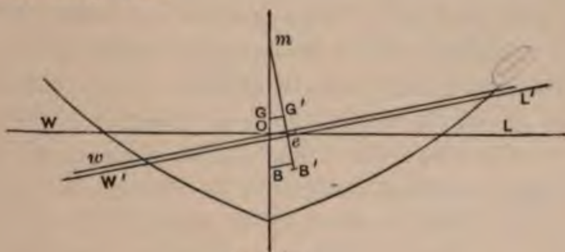


FIG. 47.

In practice it will generally be found convenient to proceed as follows in applying Barnes' method: In Fig. 47 draw an inclined water-line *wl* through some arbitrarily chosen point of the original water-line *WL*. In this case the inclined water-line is drawn through *O*, the middle of the original water-line. Determine the volumes of the wedges by aid of the integrator or by Simpson's rule or the trapezoidal rule; the volumes will, in general, be unequal. Very commonly the wedge *lOL* will be the larger; with a pronounced tumble-home above the water-line *WOw* may be the larger. Find the area of the inclined water-line *wl* from the half-breadths measured from *O*; the half-breadths are generally unequal, and the water-line is not symmetrical about a longitudinal axis through *O*. The area may be found by treating the half-breadths by Simpson's rule, or the water-line may be laid down and faired, and then measured with an integrator. Divide the difference of the volumes of the wedges *lOL* and *wOW* by the area of the inclined water-line *wl*; the result is the distance from *wl* at which a parallel water-line *W'L'* may be drawn which will cut off wedges of equal volume. Whatever error the method may have lies in the assumption that the area of the

water-line $W'L'$ is equal to that of the water-line wl; the layer is usually thin and the error insignificant. Should a layer appear to be too thick in any case, the volumes of the new wedges may be measured, and if there is an appreciable difference of volume a second approximation can be made; but this will rarely, if ever, occur in practice.

If we choose we can now proceed with the new wedges cut off by the water-line $W'L'$, and find the moments about the point e, but it will be found more expeditious to begin and carry on the calculations with reference to a longitudinal axis through O. Thus we begin by calculating the volumes of the wedges WOw and lOL and their moments about the axis through O, and find the excess of volume of the larger wedge (lOL, for example). We find also the area of the inclined water-line wl and its moment about the axis through O, and therefrom determine the distance of its centre of gravity from O. The layer $wlL'W'$ is assumed to have the same area on the two sides, and its centre of gravity will be at the same distance from O as the centre of gravity of the water-line wl. Its moment about the axis through O will be found by multiplying the distance of its centre of gravity by the excess of volume of the larger wedge, since the layer is made equal in volume to that excess. Now the moment of the wedge LeL' about the axis through O is obtained by subtracting the moment of part of the layer, namely, $lOeL'$, from the moment of the wedge lOL; and the moment of the wedge WeW' about the axis at O is obtained by adding the moment of $wOeW$ to the moment of the wedge WOw. The difference of the moments of the wedges of immersion and emersion will then be

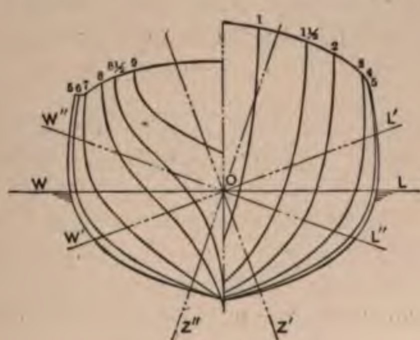
$$\begin{aligned} &\text{moment } LeL' - \text{moment } WeW' \\ &= \text{moment } lOL - \text{moment } lOeL' - (\text{moment } wOW + \text{moment } wOeW') \\ &= \text{moment } lOL - \text{moment } wOW - \text{moment } wlL'W'. \end{aligned}$$

The moment of the wedge WOw is, of course, negative, and becomes positive when subtracted algebraically. But it has already been pointed out that the difference of the moments of the equal wedges of immersion is the same whatever may be the axis chosen, consequently the calculation just outlined gives the difference of the moments of the wedges of immersion and emersion about an axis

through B, and the difference of moments is the moment of the inclined carene cut off by $W'L'$ about a longitudinal axis through B, the original centre of buoyancy. Finally, the moment of the volume of the inclined carene about the axis through B divided by the displacement of the ship in cubic feet gives the arm BN, Fig. 46, of the righting couple due to form, which is also the component of the motion of the centre of buoyancy parallel to the inclined water-line $W'L'$.

In Fig. 48 the body plan of the U. S. Light-ship No. 51 is redrawn from the lines given by Fig. 26, and to the same scale. The inclined water-line $W'L'$, drawn at an angle of 20° with the original water-line WL, defines the immersed wedge LOL' for the fore body, and the emerged wedge WOW' for the after body. The inclined water-line $W''OL''$ defines the immersed wedge WOW'' for the after body, and the emerged wedge LOL'' for the fore body. No confusion need arise from this double system of inclined water-lines, and there is considerable saving of labor in using a single-body plan, especially as there is usually such a plan at hand.

To illustrate the application of the integrator to Barnes' method there will be given the work for the U. S. Light-ship No. 51 as applied to the body plan of Fig. 48.



Scale 1/16 inch = 1 foot.

FIG. 48.

The following table gives the areas and moments of the transverse sections of the wedges of immersion and emersion as determined by tracing the sections with the integrator and applying the proper factor for the instrument and the factor for the scale of the drawing.

BARNES' METHOD.

Stations.	Immersed Wedge.		Emerged Wedge.	
	Areas, Sq. Ft.	Moments.	Areas, Sq. Ft.	Moments.
1	0.56	0.00	0.56	0.00
$1\frac{1}{2}$	5.12	18.21	3.98	6.07
2	11.94	72.82	9.67	36.40
3	25.03	194.13	20.47	145.63
4	31.00	279.13	26.16	206.31
5	32.00	285.19	27.30	230.58
6	30.25	254.86	24.46	194.28
7	23.99	182.10	17.06	109.22
8	11.37	60.68	5.12	18.20
$8\frac{1}{2}$	2.25	6.07	1.71	6.07
9	0.00	0.00	0.00	0.00

In Fig. 49 the areas and moments are laid off as ordinates at the several stations, those for the emerged wedge above and those for the immersed wedge below, and curves of areas and moments are drawn. There is also drawn on the same figure the inclined water-line with half-breadths taken from Fig. 48. The area of the curve of areas for the immersed wedge is 4.63 square inches, and that for the emerged wedge is 3.64 square inches. The scale for abscissæ is 16 feet to the inch, and the scale for the transverse area is 30 square feet to the inch. Consequently the volumes of the wedges are:

Immersed wedge...... $4.63 \times 30 \times 16 = 2222$ cubic feet
Emerged wedge....... $3.64 \times 30 \times 16 = 1756$ cubic feet

Excess of immersed wedge........ 466 cubic feet.

The inclined water-line of Fig. 49 is 8.16 square inches, and the scale of the drawing is 16 feet to the inch, so that the real area of the water-line is

$$8.16 \times 16^2 = 2089 \text{ square feet.}$$

The thickness of the correction layer is, therefore

$$466 \div 2089 = 0.22 \text{ feet.}$$

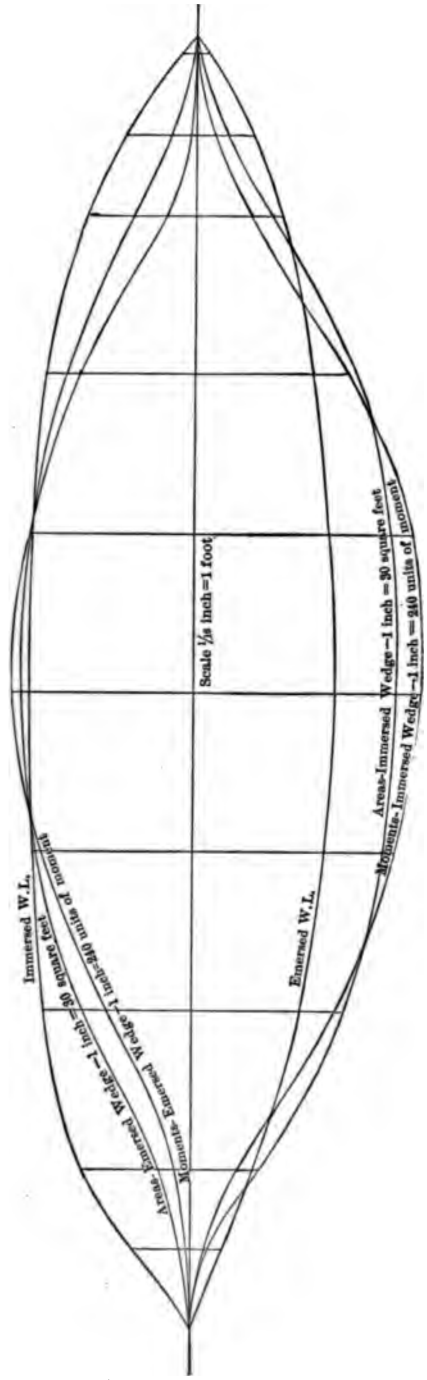

Immersed W.L.
Areas-Emersed Wedge—1 inch=30 square feet
Moments-Emersed Wedge—1 inch=240 units of moment
Scale 1/8 inch=1 foot
Emersed W.L.
Areas-Immersed Wedge—1 inch=30 square feet
Moments-Immersed Wedge—1 inch=240 units of moment

FIG. 49.

The moment of the inclined water-line in Fig. 49 is 0.69 as measured by the integrator, so that with a scale of 16 feet to the inch the moment of the real water-line about the fore-and-aft axis is

$$0.69 \times 16^3 = 2829.$$

Consequently the centre of figure of the inclined water-line is

$$\frac{2829 \text{ units of moment}}{2089 \text{ square feet}} = 1.35 \text{ feet}$$

to the right of the point O of Fig. 48 along the inclined water-line.

Now the volume of the corrective layer is equal to the excess of volume of the immersed wedge, i.e., 466 cubic feet. Consequently the moment of the correction layer is

$$466 \times 1.35 = 629.$$

The area of the curve of moments for the immersed wedge in Fig. 49 is 4.62 square inches, and for the emerged wedge is 3.32 square inches. The scale for ordinates is 240 units of moment per inch, and the abscissæ are 16 feet to the inch, so that the moments of the wedges and the stability of the ship may be computed as follows:

Moment of immersed wedge......	$4.62 \times 240 \times 16 = 17740$
Moment of emerged wedge.......	$3.32 \times 240 \times 16 = 12748$
	30488
Moment of correction layer....................	629
Moment of carene..........................	29859

The displacement of the ship (see page 34) is 11750 cubic feet, so that the arm of the righting moment due to form is

$$29859 \div 11750 = 2.54 \text{ feet}.$$

Now the centre of buoyancy of the ship is 3.4 feet below the load water-line (page 52), and if the centre of gravity of the ship is assumed to be at the load water-line, as in previous computations, the righting arm will be

$$2.54 - 3.4 \sin 20^\circ = 1.4 \text{ feet},$$

a result that may be compared with the determination of the righting arm by the method of cross-curves on page 87.

Fore-and-aft Position of Centre of Buoyancy.—It is customary to make all calculations for transverse stability as though the centre of buoyancy remained in the plane of inclination through the centre of gravity. The distance that the centre of buoyancy will move out of that plane of inclination can readily be found after the calculation of transverse stability has been made by Barnes' method, and it is customary to determine that distance as a measure of the effect of neglecting the fore-and-aft movement of the centre of buoyancy. In both this and in the main calculation by Barnes' method the ship is supposed to be inclined about a fixed longitudinal axis; an attempt to produce such an inclination of a ship floating freely, by the action of a couple in the plane of inclination through the centre of gravity, will result in giving the ship an inclination about some horizontal axis which makes a small angle with the longitudinal axis assumed for the purposes of calculation. A full discussion of this matter will be given later.

The method of procedure is to determine the moments of the wedges of immersion and emersion and of the corrective layer, referred to the plane of the midship section, and take the algebraic sum, considering distances forward to be positive and distances aft to be negative. If the resultant moment is divided by the volume of the carene, the quotient is the distance that the centre of buoyancy moves.

In the tables on page 108 the computations for determining the longitudinal movement of the centre of buoyancy is made by Simpson's rule. To find the moment of the immersed wedge each transverse section is to be multiplied by its distance in feet from the midship section, and these products summed up by Simpson's rule, treating the forward and after parts separately, will give the moments of those parts. Treating the moment of the forward part as positive and summing up algebraically gives the moment of the immersed wedge. The emerged wedge is to be treated in a similar way. It is convenient in the table to multiply the areas of transverse sections by the number of intervals, and reserve the distance between stations. The reserved distance between stations,

together with the coefficient for Simpson's rule, gives a factor for finding moments.

CALCULATION OF FORE-AND-AFT POSITION OF CENTRE OF BUOYANCY.

	Immersed Wedge.					Emerged Wedge.				
Stations.	Areas.	Simpson's Multipliers.	Functions for Areas.	Number of Intervals.	Functions for Moments.	Areas.	Simpson's Multipliers.	Functions for Areas.	Number of Intervals.	Functions for Moments.
1	0.56	½	0.28	4	1.12	0.56	½	0.28	4	1.12
1½	5.12	2	10.24	3½	35.84	3.98	2	7.96	3½	27.86
2	11.94	1½	17.91	3	53.73	9.67	1½	14.50	3	43.50
3	25.03	4	100.12	2	200.24	20.47	4	81.88	2	163.76
4	31.00	2	62.00	1	62.00	26.16	2	52.32	1	52.32
5	32.00	4	128.00	0	352.93	27.30	4	109.20	0	288.56
6	30.25	2	60.50	1	60.50	24.46	2	48.92	1	48.92
7	23.99	4	95.96	2	191.92	17.06	4	68.24	2	136.48
8	11.37	1½	17.05	3	51.15	5.12	1½	7.68	3	23.04
8½	2.27	2	4.54	3½	15.89	1.71	2	3.42	3½	11.97
9	0.00	½	0.00	4	0.00	0.00	½	0.00	4	0.00
					319.46					220.41
				Difference	33.47				Difference	68.15

AREA AND CENTRE OF GRAVITY OF INCLINED WATER-LINE.

Stations.	Whole Breadths.	Simpson's Multipliers.	Functions for Breadths	Number of Intervals.	Functions for Moments
1	2.6	½	1.3	4	5.2
1½	10.8	2	21.6	3½	75.6
2	16.4	1½	24.6	3	73.8
3	22.8	4	91.2	2	182.4
4	25.2	2	50.4	1	50.4
5	25.7	4	102.8	0	387.4
6	24.3	2	48.6	1	48.6
7	21.4	4	85.6	2	171.2
8	14.3	1½	21.5	3	64.5
8½	7.4	2	14.8	3½	51.8
9	0.0	½	0.0	4	0.0
			462.4		336.1

462.4) 51.3

.111 × 13.43 = 1.49 ft.

It is further convenient to multiply the areas in the table first by Simpson's multipliers and afterwards by the number of intervals,

thus getting the functions for moments. The area of the section at the midship station is multiplied by zero, but the place for the product is assigned to the sum for the forward part; the sum for the after part is set down below.

The sum of functions for moments for the forward part of a wedge multiplied by the reserved factor will give the moment for that part, and in like manner the moment for the after part may be found. But since the difference of these moments is to be taken, it is convenient to take the difference of the sums of functions and multiply it by the factor. Thus for the immersed wedge the difference is 33.47 and the moment is

$$33.47 \times 13.43 \times \frac{13.43}{3} = 2014,$$

where 13.43 is the reserved distance between stations and $\frac{1}{3} \times 13.43$ is the coefficient for Simpson's rule. In like manner the moment for the emerged wedge is

$$68.15 \times 13.43 \times \frac{13.43}{3} = 4097.$$

In the table for the inclined water-line there are given the whole breadths of that water-line, which are first treated by Simpson's multipliers, as in the computation for area. Then the number of intervals is introduced to get the functions for moments. The sum of the functions for areas multiplied by the coefficient for Simpson's rule will give the area of the inclined water-line, and the sum of the functions for moments multiplied in succession by the reserved interval (13.43) and by the coefficient will give the moment. But as the moment is to be divided by the area, to find the fore-and-aft location of the centre of figure of the water-line the reserved factor may be reduced to

$$\frac{13.43 \times \frac{1}{3} \times 13.43}{\frac{1}{3} \times 13.43} = 13.43.$$

The forward functions for moments are summed separately and the result is set down opposite the midship station, and the after moments are summed and the result is set down below the table. The difference (51.3) is divided by the sum of the functions for

area (462.4), and the quotient (0.111) multiplied by the factor 13.43 gives 1.49 feet for the distance of the centre of figure of the inclined water-line forward of the midship section. The volume of the corrective layer is 466 cubic feet; consequently its moment with reference to the midship section is

$$466 \times 1.49 = 694.$$

The moment of the inclined carene about an axis in the plane of inclination through the centre of gravity of the ship may be obtained by adding the moment of the immersed wedge and subtracting the moment of the emerged wedge and the moment of the corrective layer. The moment of the inclined carene is, therefore,

$$2014 - 4097 - 694 = 2772.$$

The displacement of the ship is 11750 cubic feet, so that the centre of buoyancy of the inclined carene is

$$2772 \div 11750 = 0.24$$

of a foot abaft the centre of buoyancy of the ship when erect.

Taylor's Method.—It is the habit of the Bureau of Construction and Repair of the U. S. Navy to make numerical calculations of stability from the lines of all naval vessels before their completion; and for this purpose the following method, devised by Naval Constructor D. W. Taylor, is used. The Bureau issues a book of instructions with examples and several useful tables; persons who desire to use the method, especially for government work, should apply for this book of instructions, in which are given many details that would be out of place here and which cannot be readily understood without the numerical examples. It is, however, proper and convenient to give a statement of the general principles of this method in connection with the general theory in this book.

Let Fig. 50 represent a ship with the load water-line WL and with the centre of buoyancy B. Take the middle point O of WL for a pole, and through it draw an inclined water-line $W'L'$, cutting off a carene with its centre of buoyancy at B'; the volume of the new carene is in general different from that of the original carene.

Let the volumes of the wedges of immersion and emersion be v_i and v_e, then the volume of the new carene is

$$V' = V + v_i - v_e, \quad \ldots \ldots \quad (1)$$

Let the centres of gravity of the immersed and emerged wedges be i and e, and let ih and ef be drawn perpendicular to $W'L'$; also

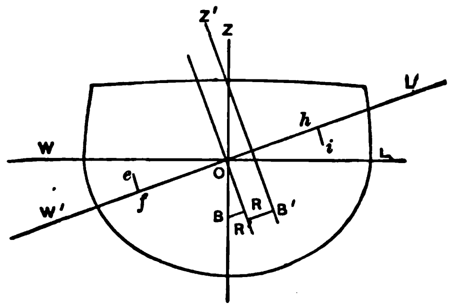


FIG. 50.

draw BR and $B'R'$ perpendicular to and OR' parallel to the new vertical $B'z'$. Taking moments about the orign O,

$$V' . B'R' = V(-BR) + v_i . Oh - v_e(-Of).$$
$$\therefore V' . B'R' = v_i . Oh + v_e . Of - V . BR. \quad \ldots \quad (2)$$

The arms BR and Of have the negative sign because they are measured to the left from O, and the moment $v_e(-Of)$ is subtracted as it belongs to the emerged wedge.

The first two terms of the right-hand side of equation (2) are the moments of the wedges, without regard to sign about O, and the third term contains the term

$$BR = OB \sin\theta = a_0 \sin\theta, \quad \ldots \ldots \quad (3)$$

if we take a_0 to represent the distance of the centre of buoyancy B of the erect carene below the arbitrary pole O. From equation (2) the following expression for the righting arm may be deduced:

$$B'R' = \frac{M_i + M_e - Va_0 \sin\theta}{V'}, \quad \ldots \ldots \quad (4)$$

where M_i and M_e are the moments of the wedges of immersion and emersion, neglecting signs.

To get the moments of the wedges compute the moments of the transverse sections of these wedges at each station by the method on page 103, and then sum up for the moments of the wedges, using the trapezoidal rule in each case. At the same time the volumes of the wedges are computed so that the volume of the inclined carene V' may be obtained.

In preparing a body plan for computing stability three water-lines are drawn, as in Fig. 51; one, WL, is drawn at the load water-line; one, $W''L''$, is taken at a draught less than that at which the ship will ever float; and the other, $W'L'$, is taken at an equal interval above the load water-line. The middle points of these water-lines, O, O', and O'', are taken as poles about which radial lines are drawn at 10° intervals and numbered each way from the vertical to 180°; the radial lines are numbered each way because a single body plan is used for measuring the

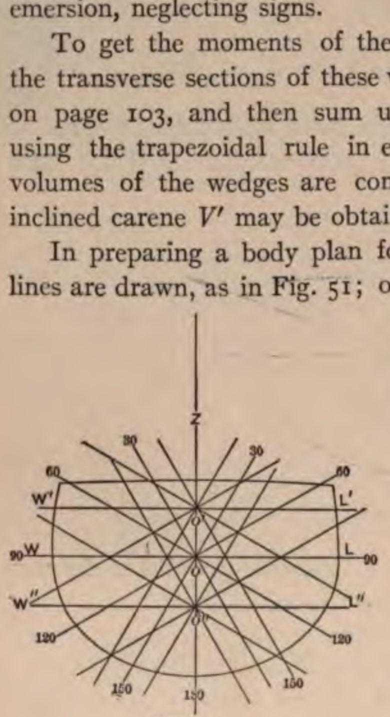


FIG. 51.

ordinates; in Fig. 51 the radial lines are drawn at 30° intervals and are numbered only about O. From the body plan as thus prepared the measurements for computing wedges may be readily made for all inclinations up to 180° if required; usually it is sufficient to carry the computations to 90° or even to 60° for steamships. The righting arms and the displacements are computed for a sufficient number of inclinations about each of the three poles, together with the directions or tangents of the angles by the method on page 90. Thus for each inclination there are obtained three points on the cross-curve, and the direction at those points, by which means the cross-curves can be sufficiently well located for all moderate inclinations. For larger angles of heel the three points of a cross-curve come near together, and at 90° they should (and in practice do nearly) coincide.

The distance of the centre of buoyancy $OB = a_0$ from any erect water-line can be taken from the curves of centres of buoyancy, Fig. 28. This quantity is important, as it gives the means of calculating the moment $Va_0 \sin\theta$ in equation (3).

As noted above, this method gives only one point on the cross-curve of stability for 90° inclination, as the three water-lines through the three poles coincide with the vertical longitudinal section through the axis of the ship. A special method is required to determine points of the cross-curve for 90° inclination. For this purpose longitudinal sections are taken as in Fig. 52, like the bow and buttock lines used in fairing the ship. The volumes cut off by these sections and the vertical locations of the centres of buoyancy are determined; for example, the centre of buoyancy is at b for the portion to the right of st, and the righting arm is Ob'; this arm is negative, as is apt to be the case for steamships.

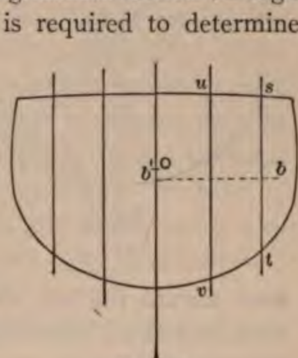


FIG. 52.

Total Immersion and Zero Displacement.—The instructions for Taylor's method give ways of determining two points for each cross-curve in addition to those found in regular order; they are for total immersion and for zero displacement. As these points can be readily located for any system for computing displacement, the manner of locating them is given separately. If $W'L'$ is any inclined water-line in Fig. 53, and B' the corresponding centre of buoyancy, then the righting arm referred to the pole O is Or let fall from O on the new vertical $B'z'$. If the water-line $W'L'$ were moved progressively farther and farther from O, then B' would approach nearer and nearer to the skin of the ship; when the displacement becomes zero the centre of buoyancy for zero displacement becomes the point of tangency B_0 of the zero displacement water-line W_0L_0.

The righting arm for total immersion is equal to $OB \sin\theta$, where B is the centre of figure of the entire ship as bounded by the skin of the ship and the water-tight deck (or decks when the ship has tight poop, forecastle, or other erection above the deck). If the

calculation for stability goes to 90°, then the point B is readily located; for the centre of figure of the whole ship is at the same height from the keel as is the centre of figure of half the ship, and the centre of figure of half the ship is the centre of buoyancy for the carene cut off at 90° inclination by the water-line passing through Oz; which centre of buoyancy is at a distance from O equal to the righting arm for that case. As the water-line approaches that one which gives total immersion its centre of gravity approaches the point like c, Fig. 53, which would be last to pass under water, and consequently cb is the proper quantity to use in equation (1), page 92. In the figure bc is measured toward the left, and consequently the angle is negative and must be laid off below the horizontal line through the corresponding point on the cross-curve.

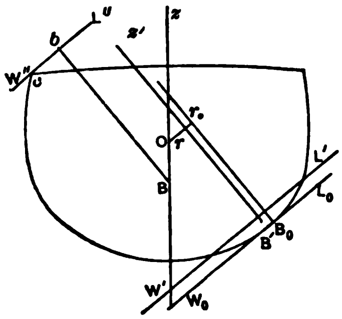


Fig. 53.

Tangent at Origin.—The form of the stability curve drawn with righting arms can be tested by drawing a tangent at the origin. For this purpose draw an ordinate at

$$180 \div \pi = 57^\circ.3,$$

and on it measure off the metacentric height to the scale used for righting arms, and from the end of this ordinate draw a line to the origin; it should be the desired tangent at the origin.

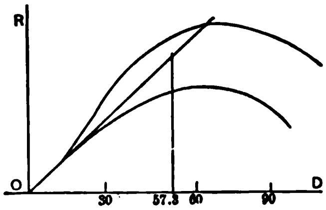


Fig. 54.

To show that this construction is correct consider the equation (2) for stability on page 74, namely,

$$\text{Righting moment} = D(h-a)\sin\theta.$$

The tangent to the angle made by a tangent at any point of the curve of stability with the horizontal axis is equal to

$$\frac{d[D(h-a)\sin\theta]}{d\theta}.$$

For small inclinations we may replace $\sin\theta$ by θ, and we may use r, the metacentric radius, instead of h, the height above B of the intersection of the new and old verticals. This gives, instead of the above differential,

$$\frac{D(r-a)d(\theta)}{d\theta}=D(r-a)$$

for the tangent of the angle at the origin. In order to use this expression as it stands we should have the curve of stability drawn with moments for ordinates, and with the angles in circular measure for abscissæ. The abscissæ are usually given in degrees, but the point selected (57°.3) is the proper unit interval for circular measure, and by properly choosing the scale for ordinates it can be made to represent either moments or righting arms; when the curves give the arms we clearly may omit the displacement from the expression, leaving $r-a$, which is the metacentric height.

Bonjean's Curves.—Thus far consideration has been given to transverse inclinations and to the methods of computing the stability for transverse inclinations. In general the metacentric method suffices for calculations of longitudinal stability and for computing changes of trim, that is, for determining longitudinal inclinations. But for certain purposes, such as the determination of changes of trim due to accidental flooding of compartments, and the investigation of launching, it is necessary to consider large longitudinal inclinations, and for such purposes a method first proposed by Bonjean is convenient.

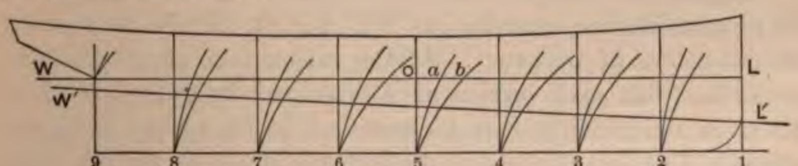


FIG. 55.

On the sheer plan, a series of water-lines is drawn with one or two above the load water-line; at each station curves of areas of transverse sections and of moments of these areas are constructed as indicated in Fig. 55; at the midship section $5a$ is the curve of transverse areas, and $5b$ is the curve of moments of transverse

areas referred to a base-line at the top of the keel. The curve of areas at the midship section has been discussed on page 48, and is drawn on Fig. 28; the other curves of areas are constructed in the same way, and will require no further discussion. The curves of moments are obtained in a similar way by finding the moment of the transverse section, up to a given water-line, about an axis at the top of the keel, and laying it off on that water-line from the vertical line in which that section is projected on the shear plan; thus *ob* is made equal to the moment about an axis at 5 of the transverse section up to the water-line *Ob*. In practice it may be found convenient to take the area and moment of half the transverse sections as represented on the body plan, and in such case computations based on Bonjean's curves refer to the half-ship and must be multiplied by 2 to give the proper results for the whole carene.

Bonjean's curves may be computed numerically from the ordinary lines of the ship provided there are enough water-lines (10 to 20); but the more convenient way is to use the integrator adjusted as usual to the load water-line; in fact this work can and should be carried on with the usual determination of displacement by aid of the integrator. With the integrator adjusted to the load water-line moments will be referred to that line as an axis, but the moments may readily be transferred to an axis at the top of the keel, as required in the statement of this method. Of course the curves could be drawn so as to give moments about the load water-line, but the curves would not be so well determined in this case.

Suppose now that we desire the displacement of the carene cut off by some inclined water-line, as $W'L'$, Fig. 55, and also the location of the centre of buoyancy. At the intersection of this inclined water-line with each section an abscissa can be drawn on which the area and moment may be measured to the proper Bonjean's curves. The areas treated by the trapezoidal rule (or any other rule of summation) will give the volume of the carene below the inclined water-line, and the moments treated in the same way will give the moment of the carene about the top of the keel, and finally the quotient obtained by dividing the moment by the volume will give the height of the centre of buoyancy above the top of the keel. The fore-and-aft location of the centre of buoyancy is found by a process

analogous to that for finding the same property for the carene below the load water-line as detailed on page 42, namely, the area at each station below the inclined water-line is multiplied by its distance from the midship section, and the moments thus obtained are summed up separately for the bow and for the stern; the difference of the moment for the bow and the moment for the stern divided by the volume of the carene gives the distance of the centre of buoyancy from the midship section.

Further consideration of this method will be given in connection with the problems to which it is applied.

Stability of Submerged Bodies.—The centre of buoyancy of a submerged body is at its centre of figure, and is not changed by any inclination of the body. For equilibrium when at rest the centre of gravity must be directly under the centre of buoyancy, and for an inclination θ, as in Fig. 56, the righting arm is Gg and the righting moment is

$$Da \sin \theta,$$

where a is the distance of the centre of gravity from the centre of buoyancy, as in our previous work.

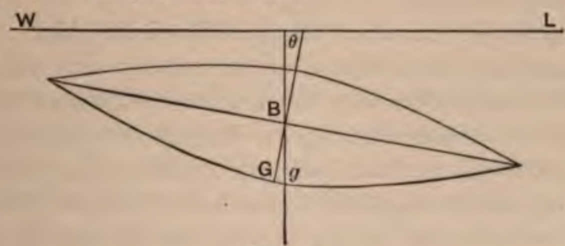


FIG. 56.

A ship floating at the surface has a very large longitudinal stability as compared with the transverse stability, but a submarine boat has exactly the same stability transversely and longitudinally, and this stability is always small even when the boat is ballasted as heavily as possible. The lack of stability of a submarine boat is one of the greatest difficulties met with in its design, more especially as the slightest change of trim when under way at once causes a change of the submersion or distance below the water.

The turning moment applied to the shaft of the propeller of a ship

which has a single screw produces a transverse inclination which, however, is too small to be easily recognized even in quiet water; but the turning moment applied to a submarine boat is liable to produce a troublesome inclination. Automobile torpedoes have two propellers, one on the shaft and one on a sleeve which has a rotation in the contrary direction, to avoid this difficulty. The trim and submersion of torpedoes are controlled by delicate automatic mechanism; the trim and submersion of a submarine boat are controlled either by such mechanism or by the steersman, who has in sight indicators which show him what the trim and submersion are at each instant.

Dynamical Stability.—The *work* required to incline a ship from a position of stable equilibrium to a given angle is called the dynamical stability.

Analysis of Dynamical Stability.—The work of inclining a ship may be analyzed into the following elements:

(1) Raising the centre of gravity of the ship.
(2) Depressing the centre of buoyancy.
(3) Wave-making.
(4) Eddy-making.
(5) Friction of the water on the skin of the ship.

The last three items depend on the velocity with which the ship is inclined, and become very small when the inclination is slow; it is customary to assume that they are insignificant and to include only the first two in the computations; if a ship is inclined rapidly, as by a sudden squall, the neglected items form a margin of safety which is truly unknown, but which appreciably reduces the danger of capsizing.

Determination of Dynamical Stability.—The customary and desirable way of determining dynamical stability is to derive it from the curve of statical stability. Taking the usual expression for the righting moment,

$$D(h-a)\sin\theta, \qquad (1)$$

and assuming an increment $\Delta\theta$ of the angle of inclination, the work required to give the ship that increment is

$$D(h-a)\sin\theta\,.\,\Delta\theta, \qquad (2)$$

and the work of inclining the ship from the position of erect equilibrium to the angle θ is

$$W=\int_0^{\theta_1} D(h-a)\sin\theta\,.\,d\theta\,. \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

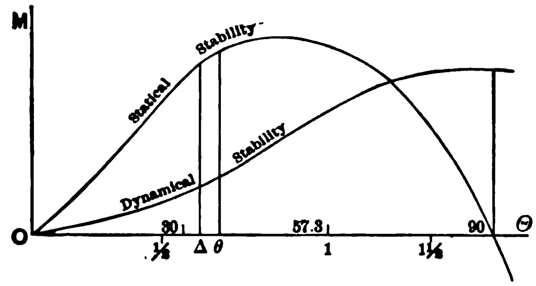


FIG. 57.

Let Fig. 57 represent a curve of statical stability with abscissæ laid off in circular measure and with righting moments laid off in foot-tons, the unit being one ton at the end of an arm one foot long. Then the expression for the righting moment in equation (3) can be replaced by the ordinate M, giving

$$W=\int_0^{\theta_1} M d\theta.$$

The curve of dynamical stability can therefore be drawn as an integral curve derived from the curve of statical stability by the usual methods. The ordinates of the curve of dynamical stability represent foot-tons of work, the unit being one ton raised one foot high. The dynamical curve can be drawn from the statical curve by the integraph, or it may be drawn point by point by measuring the area under the statical curve, allowing for scales of abscissæ and ordinates, or it may be computed by Simpson's rule.

Commonly the curve of statical stability is drawn with the inclinations in degrees and the righting arms in feet. In such case the scales of abscissæ and ordinates may be readily changed to give circular measure and righting moments, noting that the unit of circular measure comes at $180 \div \pi = 57^\circ.3$.

It is important that the factor for the scales of the drawing shall be correctly determined. As an example, we have 1.17 square

inches for the area under the curve of statical stability in Fig. 40 up to 30°. Now one inch of abscissa represents 15°, which in circular measure is

$$\frac{15\pi}{180}.$$

The vertical scale is, one inch represents 1.5 of a foot of righting arm, and the displacement of the ship (see page 37) is 336 tons. Consequently the factor for reduction is

$$\frac{15\pi}{180}\times 1.5\times 336 = 132,$$

and the dynamical stability at 30° is

$$1.17\times 132 = 154 \text{ foot-tons.}$$

It will be noted that the curve of dynamical stability rises slowly at first, that it is steepest for that angle for which the statical stability is a maximum, and that it reaches a maximum at the angle for which the statical stability becomes zero. If the curves are continued beyond that angle, the curve of statical stability lies below the axis of abscissæ, and the curve of dynamical stability, having passed its maximum, begins to descend.

Reserve of Statical Stability.—Let *Oae*, Fig. 58, represent a part of the curve of statical stability with ordinates representing the actual righting moments in foot-tons, and let the curve *dac* represent in the same manner a variable inclining moment of the wind acting on the sails. At the point of intersection of the curves the inclining moment *ab* is equal to the righting moment, so that if the ship at the inclination θ' is affected by the inclining moment *ab*, it will be in equilibrium; this may represent the condition of a ship sailing at a constant angle of heel under a steady breeze. If, however, the ship when erect is suddenly exposed to the action of an inclining couple whose moment is represented by the curve *dac*, the work done by the couple while the ship is moving from the erect position to the angle θ', will be represented by the area *Odab*. Of this work, a part, *Oab*, will be required to overcome the righting moment or stability of the ship, but the remainder, *Oda*, will be available for other work; this available work will be expended in part in overcoming the resistance

of the water (friction and wave-making), and in part in imparting velocity to the ship. The kinetic energy imparted to the ship will be expended in producing further inclination until it is used up, and the ship will come to rest at an angle greater than θ', for which the righting moment is greater than the inclining moment; the ship will then roll back toward the erect position and then return, and finally will come to rest at the angle θ', where the inclining and righting moments are equal.

If we neglect the work expended in overcoming resistance (friction and wave-making), we may readily determine the angle to which the ship will roll under the influence of the inclining moment represented by the curve *dac*; i.e., find by trial an ordinate *ec* which will make the area *aec* equal to the area *Oad*, then the excess of work required to produce the inclination from θ' to θ'' over the work of the inclining couple will be equal to the excess of work of the inclining couple for the angle $O\theta'$, and at θ'' the ship will come to rest. The effect of resistance is, of course, to bring the ship to rest at an angle less than θ'', in proportion as the inclination is more rapid.

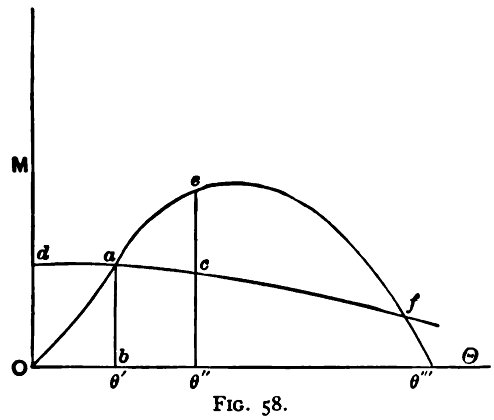


FIG. 58.

The area *aefc* between the curves of stability and of inclining moments is called the reserve of dynamical stability; it represents the work that must be done by some additional source of energy upon the ship while sailing under a steady breeze to bring it to the point *f* at which the ship is liable to capsize, for at that point the ship

is in unstable equilibrium, and beyond that point the inclining moment is greater than the righting moment. The reserve of stability gives some idea of the ability which a ship may have when

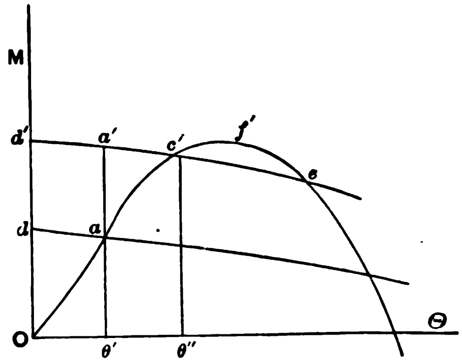


Fig. 59.

inclined under sail to endure an additional effort, like that of a sudden squall.

It has been seen that the ship when at rest and erect may be inclined by the sudden action of a breeze to an angle greater than θ',

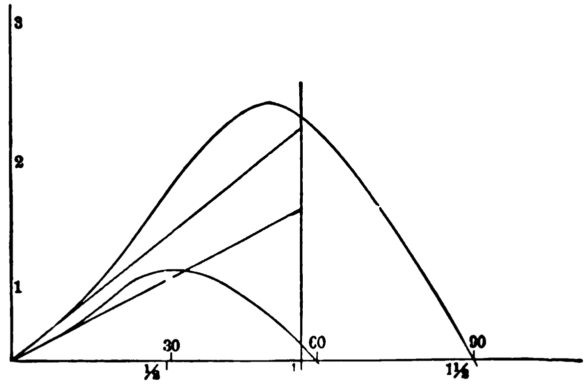


Fig. 60.

where the curves of stability and inclining moments intersect; neglecting resistance, the angle is θ'', making $aec = Oda$. Should it happen that Oda, Fig. 58, is greater than the reserve of stability $aofc$, then the ship may be capsized by the sudden action of a breeze which it could endure if applied gradually. In much the same way

a ship when inclined to an angle θ', as represented by Fig. 59, may be placed in jeopardy by a sudden increase in the intensity of the wind, as represented by dd', if the area $a'c'a$ is greater than $c'f'e$.

There are two distinct types of curves of stability, those for sailing-ships and those for steamers; these may be represented by the curves so marked on Fig. 60. The sailing-ship usually has a greater metacentric height and has high sides, so that the curve of statical stability extends to 90° or beyond; the curve for a steamer may extend to about 60° and yet the stability may be ample, for the steamer carries little or no sail; in order that a steamer may be steady in a seaway the stability must not be excessive, as it leads to quick and violent rolling which may be very unpleasant or even dangerous.

CHAPTER IV.

SURFACES OF BUOYANCY AND OF WATER-LINES.

THE discussion of stability in the preceding chapter is that commonly given, and is allowed to stand as a basis for the methods of computation used in practice. Strictly it should be limited to that case on which the demonstration is based, namely, a ship that is symmetrical fore and aft as well as transversely, and the attempt to determine the fore-and-aft motion of the centre of buoyancy which is made in connection with Barnes' method is illogical and incorrect. Since large ships are not very unsymmetrical, the error of treating them as though they were symmetrical is not important, and the treatment in connection with Barnes' method may be looked upon as an approximate correction of a small error.

This chapter, which is purposely disconnected from the general discussion of stability and methods of computation, gives a fundamental and logical treatment of the most general problems connected with stability. It gives a broader view and firmer grasp of the subject, and will give a better appreciation of some of the applications of stability, especially to such problems as adding large weights or breaking open compartments to the sea. Some of the definitions of the preceding chapters will be repeated and given a wider application. The propositions to be established will be treated as problems of geometry with the aid of the methods of infinitesimal calculus.

If the plane of the free surface of the water be imagined to be produced through a floating body, the figure which is cut off by this plane, and which is bounded by the wetted surface of the body and the produced plane, is called the carene. If the weight of the body is constant, the volume of the carene will also be constant; the term isocarene will be applied to carenes which have a constant

volume, and also to all properties of such carenes, as for example to the plane or water-line which cuts them off.

The centre of figure of the carene is the centre of buoyancy of the floating body to which it belongs. By varying the axis and the angle of inclination a floating body may be turned to all possible positions. Horizontal axes of inclination only will be considered, for a rotation of the floating body around a vertical axis changes its aspect, but does not change the form of the carene. Any plane perpendicular to the axis of inclination may be taken as the plane of inclination, and the body and its sections and curves may be projected on it.

Each position of the floating body will have its own centre of buoyancy, and all the centres of buoyancy will lie on some surface which will be called the *surface of buoyancy*. The surface of buoyancy for a given floating body is a definite figure which depends only on the contour of the body; as the body is turned the surface of buoyancy turns with it. It is convenient to draw diagrams of the surface of buoyancy and of its sections and projections, as though the floating body remained at rest and the water-line were shifted so as to cut off the carenes under consideration. When this convention is followed there is a plane or water-line corresponding to each carene; a surface which touches all the isocarene water-lines, and which is consequently an envelope of them, is called the *surface of water-lines*.

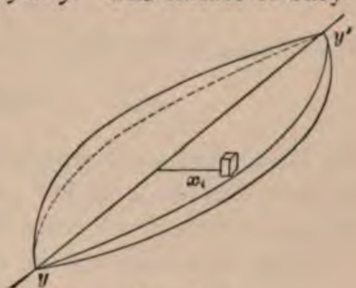


FIG. 61.

Intersection of Isocarene Water-lines.—As the angle between two isocarene water-lines approaches zero their intersection approaches a line, through the centre of gravity of either of them, which is parallel to the axis of inclination.

If the water-lines did not intersect, they would be parallel and would cut off unequal carenes. There may be two isocarene water-lines parallel to each other at 180°, but that fact does not affect the preceding statement.

Consider two isocarene water-lines which make a small angle with

each other and intersect in the line yy', as in Fig. 61. In one of the planes take a small area

$$\Delta x \Delta y = \Delta A,$$

and on it erect a parallelopiped having the height $x_i \Delta\theta$, where x_i is its distance from the intersection yy', and $\Delta\theta$ is the angle between the planes. The volume of the parallelopiped is

$$x_i \Delta\theta \, \Delta A,$$

and the volume of the immersed wedge is

$$v_i = \Delta\theta \int x_i dA,$$

while the volume of the emerged wedge is

$$v_e = \Delta\theta \int x_e dA.$$

The volumes of the wedges must be equal, consequently

$$\int x_i dA = \int x_e dA;$$

but these expressions represent the moments of the two sides of the water-line about the line yy', and the moments being equal, that line passes through the centre of gravity of the water-line.

Either water-line can be taken as the original and the other as an isocarene water-line after an inclination of $\Delta\theta$; consequently the intersection passes through the centre of gravity of either of them.

Each water-line is parallel to the axis of inclination, consequently their intersection is parallel to that axis, and so also is the line yy' to which the intersection approaches as $\Delta\theta$ approaches zero.

Coordinates of the Centre of Buoyancy after a Small Inclination.— In Fig. 62 let B be the original centre of buoyancy of a floating body, and let B' be the new centre of buoyancy after a small inclination $\Delta\theta$. Through B, as an origin, draw BY parallel to the axis of inclination, BZ vertically upwards, and BX perpendicular to the other two axes: a plane through BX and BZ will be perpendicular to the axis of inclination, and will cut a section from the body as shown. The original water-line cuts the plane of inclination in WL, and the isocarene water-line cuts it in $W'L'$. Through O, the

trace of the vertical BZ on the water-line WL, draw an axis OY parallel to BY; it may be taken for the axis of inclination.

If $\Delta\theta$ is assumed to approach zero, the intersection of the two water-lines gY will approach a line parallel to the axis of inclination through the centre of gravity g of the water-line WL. Take g as the origin of coordinates in the water-line WL, and draw the axis gX parallel to BX; the two systems of coordinates, one in the plane WL

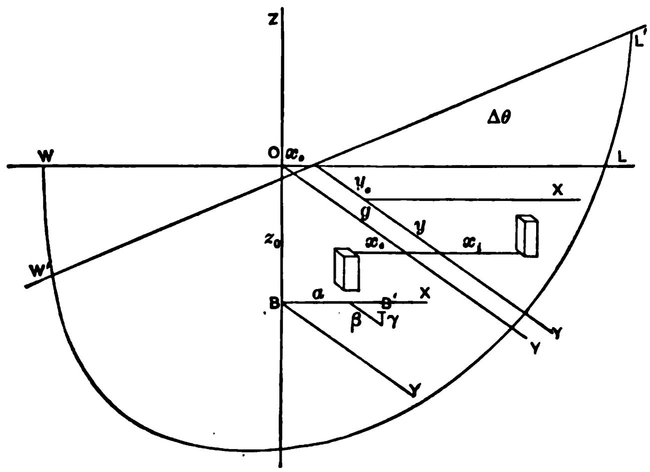


FIG. 62.

with the origin at g and the other in space with the origin at B, have their axes parallel and need not lead to confusion. The coordinates of the point O referred to the axes in the plane WL are $-x_0$ and $-y_0$, as shown in Fig. 62.

It is required to find the coordinates α, β, and γ of the new centre of buoyancy referred to the origin B; if $\Delta\theta$ is infinitesimal, so also will be these coordinates. To obtain these coordinates it is sufficient to take moments of the new carene with regard to each of the coordinate planes; noting that the moment of the new carene with regard to any plane through the original centre of buoyancy B can be obtained by taking the moment of the wedge of immersion and subtracting the moment of the wedge of emersion with regard to that plane.

Consider a small parallelopiped whose base has the area

$\Delta x \Delta y = \Delta A$ in the plane WL, and has the coordinates x_i and y referred to the origin at g. If the parallelopiped is bounded at the top by the plane $W'L'$, its height will be $x_i \Delta\theta$, and its volume will be $x_i \Delta\theta \Delta A$, so that its moment referred to the plane OBY may be written

$$(x_i + x_0)x_i \Delta\theta \Delta A.$$

The moment of the entire wedge of immersion is

$$\Delta\theta \int (x_i + x_0)x_i dA = \Delta\theta \int x_i^2 dA + x_0 \Delta\theta \int x_i dA. \quad . \quad . \quad . \quad (1)$$

In like manner the moment of the emerged wedge is

$$-\Delta\theta \int (x_e - x_0)x_e dA = -\Delta\theta \int x_e^2 dA + x_0 \Delta\theta \int x_e dA; \quad . \quad . \quad (2)$$

the negative sign being used, as the arm is measured to the left.

Subtracting the moment of the emerged wedge from the moment of the immersed wedge, the moment of the new carene is

$$V\alpha = \Delta\theta \left\{ \int x_i^2 dA + \int x_e^2 dA \right\} + x_0 \Delta\theta \left\{ \int x_i dA - \int x_e dA \right\}, \quad . \quad (3)$$

where V is the volume of the isocarene. But the moment of the water-line WL about the axis gY is zero, and consequently the second term of the right-hand member vanishes. The first term represents the moment of inertia of the water-line WL about gY, which may be written I, and equation (3) gives

$$V\alpha = I\Delta\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$\alpha = \frac{I}{V}\Delta\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The moment of the parallelopiped with reference to the plane OBX is

$$(y + y_0)x_i \Delta\theta \Delta A,$$

and the moment of the entire wedge of immersion is

$$\Delta\theta \int (y + y_0)x_i dA = \Delta\theta \int x_i y dA + y_0 \Delta\theta \int x_i dA, \quad . \quad . \quad (6)$$

and the moment of the emerged wedge is

$$\Delta\theta \int (y + y_0)x_e dA = \Delta\theta \int x_e y dA + y_0 \Delta\theta \int x_e dA; \quad . \quad . \quad . \quad (7)$$

so that the moment of the new carene is

$$V\beta = \Delta\theta\left\{\int x_i y dA - \int x_e y dA\right\} + y_0 \Delta\theta\left\{\int x_i dA - \int x_e dA\right\}; \qquad (9)$$

where the last term is again equal to zero; the first term is the moment of deviation of the water-line with regard to the axes gX and gY and may be represented by K. Consequently

$$V\beta = K\Delta\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

or

$$\beta = \frac{K}{V}\Delta\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

The moment of the parallelopiped with reference to the plane XBY is

$$(z_0 + \tfrac{1}{2}x_i\Delta\theta)x_i\Delta\theta\Delta A,$$

and the moment of the immersed wedge is

$$\Delta\theta\int(z_0 + \tfrac{1}{2}x_i\Delta\theta)x_i dA = z_0\Delta\theta\int x_i dA + \frac{\overline{\Delta\theta}^2}{2}\int x_i^2 dA, \quad . \qquad (12)$$

while the moment of the emerged wedge is

$$\Delta\theta\int(z_0 - \tfrac{1}{2}x_e\Delta\theta)x_e dA = z_0\Delta\theta\int x_e dA - \frac{\overline{\Delta\theta}^2}{2}\int x_e^2 dA, \quad . \qquad (13)$$

so that the moment of the new carene is

$$V\gamma = z_0\Delta\theta\left\{\int x_i dA - \int x_e dA\right\} + \frac{\overline{\Delta\theta}^2}{2}\left\{\int x_i^2 dA + \int x_e^2 dA\right\}; \quad (14)$$

the first parenthesis is equal to zero and the second is the moment of inertia of the water-line, so that

$$V\gamma = \frac{\overline{\Delta\theta}^2}{2}I, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

or

$$\gamma = \frac{I}{V}\frac{\overline{\Delta\theta}^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Since $\Delta\theta$ is an infinitesimal, α and β are infinitesimals; but γ is an infinitesimal of the second order; that is, γ is very small compared with α or β.

Normal to the Surface of Buoyancy.—In Fig. 63 let B be the original centre of buoyancy and B' the new centre of buoyancy after a very small inclination $\Delta\theta$, the coordinates of B' being α, β, and γ, of which the first two are infinitesimals of the first order and the third is of the second order. A line joining B and B' will be a secant line which cuts the surface of buoyancy in those two points. If B be joined to the foot of the perpendicular from B' to the plane XBY, the small angle $\Delta\phi$ which the secant makes with the plane will be measured by

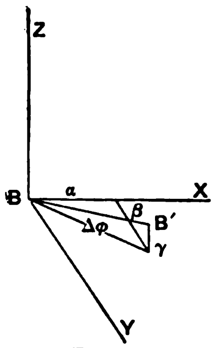


FIG. 63.

$$\tan \Delta\phi = \frac{\gamma}{\sqrt{\alpha^2+\beta^2}},$$

which is an infinitesimal, for γ is of the second order, while α and β are of the first order. When $\Delta\theta$ approaches zero B' approaches B, and BB' approaches a tangent line to the surface of buoyancy at B; at the same time the line BB' approaches coincidence with the plane XBY.

If now another axis of inclination be taken, we shall get another new centre of buoyancy B'', and the line through B and B'', as the angle of inclination approaches zero, will approach another tangent line to the surface of buoyancy at B, and at the same time it will approach the plane XBY, which is perpendicular to the original vertical at B. Thus we get two tangents at the point B to the surface of buoyancy, both of which lie in a plane through B, which plane is perpendicular to the original vertical at B.

Consequently the vertical through any point B of the surface of buoyancy of a floating body is perpendicular to the tangent plane at that point, and the tangent plane is parallel to the water-line.

Positions of Stability.—In order that a floating body may be in equilibrium, the vertical through the centre of buoyancy must pass through the centre of gravity; but the vertical through the centre of buoyancy is normal to the surface of buoyancy; therefore in a position of equilibrium the line connecting the centre of gravity to the centre of buoyancy is a normal to the surface of buoyancy.

There will be as many positions of equilibrium as there are normals from the centre of gravity to the surface of buoyancy. Commonly only two positions of equilibrium will be stable, and sometimes only one.

Form of the Surface of Buoyancy.—In the expression for the vertical coordinate of the new surface of buoyancy,

$$\gamma = \frac{I}{V} \frac{\overline{\Delta\theta}^2}{2},$$

the terms I and V are always positive, and $\Delta\theta$ appears as the square; consequently γ is always positive; hence the surface of buoyancy is all on one side of the tangent plane at any point; consequently the surface of buoyancy is everywhere convex.

If the surface of the floating body is continuous and closed, the surface of buoyancy is so also; for a vertical may be drawn through any point of the surface of the body, and that vertical must pass through a centre of buoyancy.

The surface of buoyancy for a ship is entirely within the skin of the ship and her deck. The surface of buoyancy of a catamaran will be partly within and partly outside of the two hulls. The surface of buoyancy of a circular life-buoy may be entirely outside of the buoy.

Direction of Motion of the Centre of Buoyancy.—A vertical plane through the original centre of buoyancy B and the new centre of buoyancy B', after a very small inclination, makes with the plane of inclination through B the angle ω (Fig. 64) determined by the equation

$$\tan \omega = \frac{\beta}{\alpha} = \frac{K}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The moment of deviation of the water-line is zero if that figure is symmetrical about the axis gY or about the axis gX (Fig. 62). For either condition the centre of buoyancy will start to move in the plane of inclination, though for a finite inclination it may leave that plane. If the carene is symmetrical with regard to the plane of inclination, the centre of buoyancy will remain in that plane for all inclinations; but that is a different proposition.

Curve of Buoyancy.—In Fig. 65 let B' be a new centre of buoyancy, after a finite inclination θ, about an axis parallel to By. The point B' will, during the inclination, trace a continuous curve BB' on the surface of buoyancy; which path is in general a curve of double curvature. To make this evident it is sufficient to consider that the coordinates of the point B', Fig. 65, depend on the moments of the carene with regard to the planes YBX, YBZ, and XBZ, while the coordinates α, β, and γ of a new centre of buoyancy after an

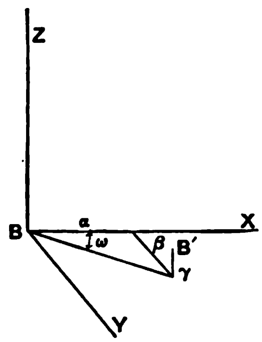


Fig. 64.

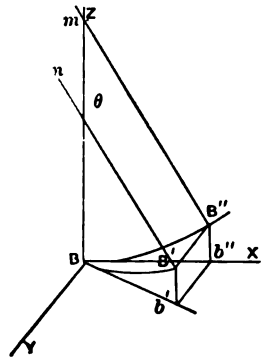


Fig. 65.

infinitesimal inclination depend on the moment of inertia and moment of deviation of the water-line; since there are no necessary relations between these sets of properties there is no necessary relation between the sets of coordinates depending on them. Consequently it cannot be shown that the centre of buoyancy will in general remain in the plane ZBB', Fig. 65, and in general it will not do so.

The projection of the path of the centre of buoyancy on to the plane of inclination is called the curve of buoyancy; as, for example, BB'', Fig. 65.

Since a ship is symmetrical transversely, the centre of buoyancy remains in the plane of inclination for changes of trim. But as a ship is not usually symmetrical fore and aft, the centre of buoyancy will leave the plane of inclination for large transverse inclinations. If the ship is initially erect, the water-line will be symmetrical about its fore-and-aft axis, and the centre of buoyancy, for small inclina-

tions, will begin to move in the plane of inclination, and will in no case depart very far from it.

Projecting Cylinder.—The tangent plane to the surface of buoyancy (Fig. 65) is parallel to the corresponding water-line, which, in turn, is parallel to the axis of inclination, and consequently the tangent plane is perpendicular to the plane of inclination and contains the projecting line $B'B''$. Hence the projecting line is tangent to the surface of buoyancy at the point B'. A series of lines projecting successive positions of the centre of buoyancy on to the plane of inclination will be the elements of a cylinder which is tangent to the surface of buoyancy, and which projects the path of the centre of buoyancy into the curve of buoyancy. If the floating body is closed and makes a complete revolution about a given axis of inclination, the path of the centre of buoyancy, the projecting cylinder, and the curve of buoyancy will all be closed.

Normal to the Curve of Buoyancy.—The normal $B'n$ (Fig. 66) to the surface of buoyancy at the point B' is perpendicular to the tangent plane at B'; that tangent plane is also tangent to the projecting cylinder along the element $B'B''$; a plane passed through $B'n$ and $B'B''$ will be perpendicular to the tangent plane, and will cut from the plane of inclination a line $B''m$ which is normal to the projecting cylinder and to the curve of buoyancy BB''.

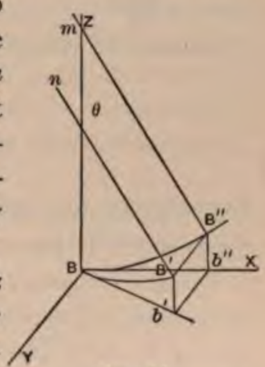


FIG. 66

Metacentre. — If the original vertical Bz (Fig. 66) passes through the centre of gravity so that the floating body is originally in equilibrium, then Bm is the term h which appears in the usual equation for statical stability,

$$D(h-a)\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in which D is the displacement in tons, a is the distance which the centre of gravity is above the original centre of buoyancy, and θ is the angle of inclination. If the angle of inclination approaches zero, the intersection m of the original vertical and the projection of the new vertical approaches a fixed limit called the metacentre.

Radius of Curvature of the Curve of Buoyancy.—If the angle of inclination is very small so that θ of Fig. 66 becomes $\Delta\theta$ of Fig. 67,

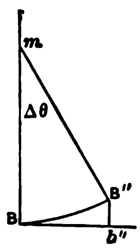


FIG. 67.

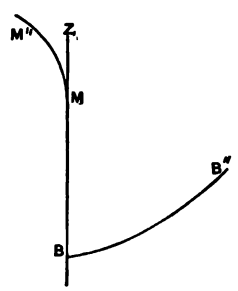


FIG. 68.

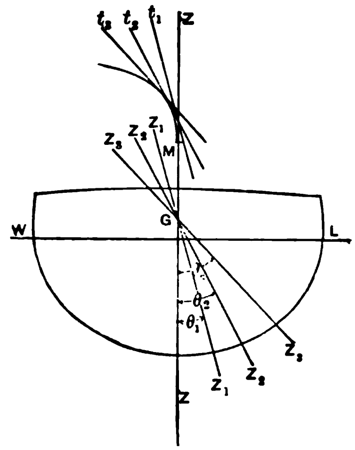


FIG. 69.

the arc BB'' may be considered to be circular and equal to $Bb''=\alpha$. The radius of curvature at the point B is then

$$Bm=\frac{\alpha}{\Delta\theta}.$$

Representing this radius of curvature by ρ, and substituting for α from equation (5),

$$\rho=\frac{I}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If the axis of inclination is parallel to that axis of the water-line which corresponds to the minimum moment of inertia, the radius of curvature of the curve of buoyancy also becomes a minimum and may be written

$$r=\frac{i}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If, on the contrary, the axis of inclination is parallel to that axis of the water-line which corresponds to the maximum moment of

inertia, the radius of the curve of buoyancy becomes a maximum and may be written

$$R=\frac{I}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

The value of the maximum and minimum radii of curvature R and r have already been deduced for doubly symmetrical bodies on page 76.

Metacentric.—The locus of the ends of the radii of curvature of the curve of buoyancy is called the metacentric (Fig. 68). This curve is the evolute of the curve of buoyancy, and, conversely, the curve of buoyancy is the involute of the metacentric.

The metacentric may be readily drawn from the curve of statical stability in the following way: Through the centre of gravity G, Fig. 69, draw a new vertical Z_1Z_1, making the angle θ_1 with the original vertical ZZ. From the curve of statical stability find the righting arm at the angle θ_1, and draw the line t_1t_1, parallel to Z_1Z_1, and at a distance equal to the length of the righting arm. In a similar way draw other lines, Z_2Z_2, Z_3Z_3, etc., and parallel lines, t_2t_2, t_3t_3, etc., at distances equal to the corresponding righting arms. The envelope of the lines t_1t_1, t_2t_2, t_3t_3, etc., will be the metacentric, which will of course pass through the metacentre. This method is subject to the criticism that the construction of the envelope is somewhat indefinite.

If exact locations of the curve of buoyancy and the metacentric are desired, they may be constructed by adaptations of the method of cross-curves. On Fig. 70 draw a series of water-lines like WL and wl as for the usual method of determining stability, and draw also a series of inclined water-lines like $W'L'$ and $w'l'$. With the integrator adjusted to the new vertical $Z'Z'$ measure the areas and moments of the transverse section of the carene bounded by a given inclined water-line (such as $W'L'$) and therefrom deduce the distance OT to the vertical $B'T$ through the new centre of buoyancy B'. The same process applied to the carene bounded by other water-lines (like $w'l'$) and continued for a series of inclinations will give the means of determining the coordinate OT for all angles and all displacements. It is apparent that OT is the righting arm of stability, and that the usual cross-curves enable us to determine OT directly.

To determine the other coordinate at TB' let the integrator be adjusted to the inclined water-line $W'L'$, and let areas and moments of transverse section be traced as before, thus giving the means of drawing another set of cross-curves from which TB' can be interpolated for any angle and any displacement.

Having the two sets of cross-curves, we may readily interpolate for the coordinates OT and TB' for the proper displacement and

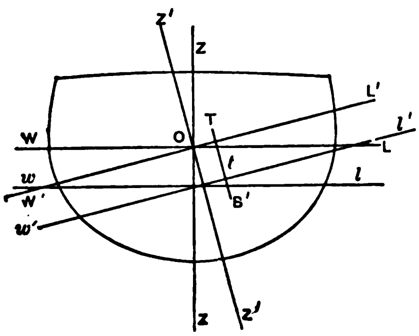


FIG. 70.

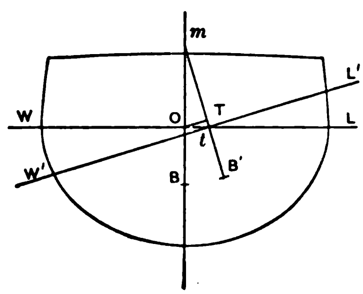


FIG. 71.

for a series of inclinations, and may plot a series of locations of B' as indicated on Fig. 71.

To complete the construction and draw the metacentre it is necessary to determine the radius of curvature ρ for each angle used in locating the curve of buoyancy by aid of equation (19),

$$\rho = \frac{I}{V}.$$

The volume V is of course the constant of the isocarene and is therefore known, but it is necessary to locate the water-line $W'L'$ correctly in order that its moment of inertia I can be determined. The most convenient way of doing this is to draw a third series of cross-curves for values of Tt, Fig. 70, at all angles and displacements; this work can be carried out at the same time that the cross-curves for OT and for TB' are constructed. Having the value of ρ, it may be laid off from B', thus giving a point of the metacentric; after a sufficient number of points are located that curve can be drawn definitely.

Sections of the Surface of Buoyancy.—The path of the centre of buoyancy for a very small inclination may be assumed to lie in a plane passed through the original vertical and the new centre of buoyancy B', Fig. 72. The curve BB' may be assumed to be the arc of a circle having its centre on the original vertical BZ. Let Fig. 73 represent the arc BB' revolved into the plane of the paper and extended to half a circle. The half-chord AB' has the length

$$\sqrt{\alpha^2+\beta^2},$$

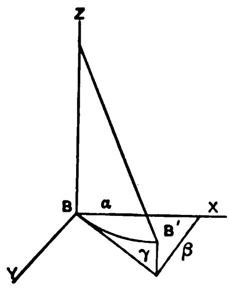


FIG. 72.

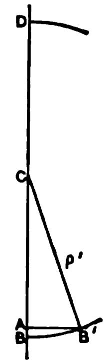


FIG. 73.

and the segment AB of the diameter BD has the length γ, while the section AD has the length $2\rho_1-\gamma$, where ρ_1 is the radius of the circle in Fig. 73, and may be taken for the radius of curvature of the section of the surface of buoyancy at B, Fig. 72, made by the plane ZBB'. Since the half-chord is a mean proportional between the segments of the diameter,

$$(2\rho_1-\gamma)\gamma=\alpha^2+\beta^2;$$

but γ is very small compared with $2\rho_1$, consequently

$$\rho_1=\frac{\alpha^2+\beta^2}{2\gamma}=\frac{1}{V}\frac{I^2+K^2}{I} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

after substituting values of α, β, and γ from equations (5), (10), and (16), and reducing. Again, introducing the radius of curvature of the curve of buoyancy by aid of equation (19),

$$\rho_1=\rho+\frac{1}{\rho}\frac{K^2}{V^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Principal Sections of Surface of Buoyancy.—Let. Fig. 74 represent the water-line which cuts off a carene whose centre of buoyancy is at a point B of the surface of buoyancy; the water-line may

have any form, symmetrical or unsymmetrical, and the point B may or may not correspond to a position of equilibrium. Through G, the centre of gravity of the water-line, draw an axis GY parallel to the axis of inclination, and GX perpendicular to GY. An infinitesimal inclination about the axis of inclination will give a new location B' on the surface of buoyancy whose coordinates α, β, and γ have been made to depend on I and K, the moment of inertia and moment of deviation of the water-line referred to the axis GX and GY.

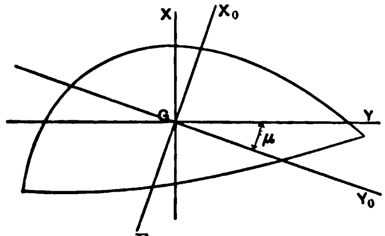


FIG. 74.

A plane passed through the original vertical BZ and B' will cut a section from the surface of buoyancy whose radius of curvature is given in terms of I and K by equation (22). Suppose that the axis of inclination be swung round, while the floating body and the points B and G remain fixed, beginning at any position of GY, for example GY_0, and take this position as an axis of reference. As the axis of inclination swings round and the coordinate axes GX and GY with it, the values of I and K will change, and the coordinates α, β, and γ and the radius of curvature ρ will vary with I and K. If I_0 and K_0 are the moment of inertia and moment of deviation of the water-line referred to GY_0 and GX_0 (perpendicular to GY_0), and if μ is the angle which GY makes with GY_0, then I and K can be expressed in terms of I_0 and K_0 and μ. To find the maximum and minimum values of ρ_1 it is sufficient to differentiate equation (22) with regard to μ, and to equate the first differential coefficient to zero, bearing in mind that

$$\frac{dI}{d\mu} = 2K;$$

this can be found in any text-book on applied mechanics.*

By this method it appears that

$$\frac{d\rho_1}{d\mu} = \frac{1}{V}\left(2K + \frac{2IK\frac{dK}{d\mu} - 2K^3}{I^2}\right) = 0; \quad . \quad . \quad . \quad . \quad (24)$$

* Applied Mechanics, Lanza, page 114.

and it is clear that it is sufficient to make K equal to zero in order to satisfy equation (24). But this is the condition that obtains when I is a maximum or a minimum; and the axis about which I is a maximum is at right angles with that for which it is a minimum. Therefore, as the axis GY swings round it will come to a position GY_0, for which K is zero and I is a minimum; at right angles to GY_0, that is GX_0, is the position of the axis about which I is a maximum. If the axis of inclination be taken parallel to GY_0, then the radius of curvature of the normal section through B will be a minimum, but if the axis of inclination be taken perpendicular to GY_0 (parallel to GX_0), then the radius will be a maximum. The axes about which the moment of inertia is a maximum or a minimum are called principal axes, and the sections which have the maximum and minimum radii of curvature are called principal sections.

The following conclusions may be drawn:

1. The principal sections at a given point of the surface of buoyancy are parallel to the principal axes of the corresponding water-line, and are at right angles with each other.

2. When a floating body is inclined about an axis which is parallel to a principal axis through the centre of gravity of the water-line, the centre of buoyancy begins to move in the plane of inclination (because K and β are then equal to zero), and the plane of inclination is one of the principal sections of the surface of buoyancy.

3. For the above condition the radius of curvature of the principal section mentioned is equal to the radius of curvature of the curve of buoyancy at the centre of buoyancy. (See equation (23).)

4. If the ship is erect, the principal sections of the surface of buoyancy are cut by fore-and-aft and transverse planes through the centre of buoyancy. The radii of curvature of these sections are

$$r=\frac{i}{V} \text{ and } R=\frac{I}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Analysis of the Righting Couple.—From the preceding investigation it appears that in general the inclination of a floating body about any axis will cause the centre of buoyancy to move to

some point as B', Fig. 75, outside of the plane of inclination. Starting from a position of equilibrium and representing the height of the metacentre above the centre of buoyancy by h, while the height of the centre of gravity is a, and representing the distance of the new centre of buoyancy from the plane of inclination through the original centre of buoyancy by b, the entire righting moment may be resolved into transverse and fore-and-aft components

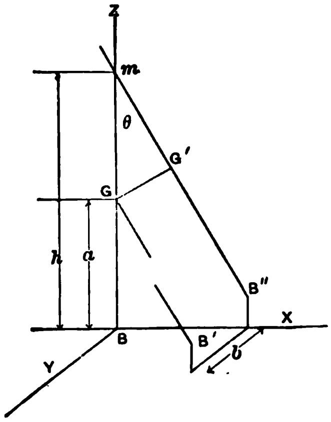


FIG. 75.

$$D(h-a)\sin\theta \quad \text{and} \quad Db. \quad . \quad . \quad . \quad (26)$$

The second component is small for ordinary inclinations of well-formed ships, and is usually ignored in computations of stability. When Barnes' method is used it is customary to find the centre of gravity of immersed and emerged wedges; the fore-and-aft distance between these centres multiplied by the displacement of one wedge gives the fore-and-aft righting moment. When any account is taken of this moment it is customary to calculate the change of trim due to that moment as though it were a longitudinal inclining moment. For example, if the centre of buoyancy is thrown forward in consequence of a transverse inclination, the computation is made of the amount that the ship will trim by the stern. Since computations of stability are valuable for comparison, and since great refinement is unnecessary, this method is entirely sufficient for practice; but it is confusing to the student when it is first presented, for, starting with a certain transverse inclining couple, it is assumed that its entire moment is employed in producing a transverse inclination, and immediately afterward it appears that there is a moment to change trim without any assigned source. The facts are, of course, that to hold the ship in the place assigned for the calculation there will be required two moments, one transverse and one fore-and-aft, equal and opposite to the righting moments represented by expressions (26), and that the total inclining moment will be equal and opposite to the resultant of the two righting moments.

Elements of Surface Stability.—For slight inclinations the righting moment is

$$D(r-a)\theta,$$

in which D is the displacement, a is the distance of the centre of buoyancy below the centre of gravity, and r is the distance of the centre of buoyancy below the metacentre. It is interesting to note the changes produced in the righting moment by changes in the form and dimensions of ships.

The metacentric height is seldom less than $1\frac{1}{2}$ feet and seldom more than 5 feet, unless it be in special forms, such as low free-board monitors, which sometimes have a metacentric height of 13 feet. Some large and heavy armor-clads have a metacentric height of $3\frac{1}{2}$ or 4 feet, but in general steamships with auxiliary sail-power have less metacentric height. It appears, then, that the metacentric height for steamships is somewhere near the same for all steamships, whatever their size; consequently we have the righting couple proportional to the displacement, or nearly so.

The metacentric height may be controlled by (1) varying the proportion of beam to length, or (2) by arranging the weights carried by the ship. The first determines the value of r, and the second of a. Now

$$r=\frac{i}{V},$$

in which i is the moment of inertia of the water-line about a transverse axis through its centre of gravity, and V is the volume of the carene. If the water-line is a rectangle having the length L and the beam b, we have

$$i=\frac{Lb^3}{12},$$

and for any form we may write

$$i=cLb^3,$$

in which c is a constant depending on the form of the water-line, but which is the same for similar forms. Again, we may make

$$V=kLbd,$$

in which d is the draught, and k is the coefficient of fineness. Replacing i and V by these values, we have

$$r=\frac{cLb^3}{kLbd}=\frac{c}{k}\frac{b^2}{d}.$$

If the form of the ship be varied in such a manner that c and k remain constant, it is clear that the value of r will vary with the square of the beam, and inversely as the draught. In general, a change of proportion will affect the constants c and k, but not to a marked degree, for any changes that are liable to be made in modelling one ship after another; the notable effect of increase in beam should be borne in mind. If two ships are quite similar, then c and k will be the same for both, and further, b will be proportional to d, so that we shall have

$$r=nb,$$

in which n is a constant depending on the form of the ship. In such case it is clear that r varies as any linear dimensions, as should be the case, since it is a linear dimension itself.

If we consider the longitudinal stability, we have for the righting moment for small inclinations

$$D(R-A)\theta,$$

in which A is the distance of the centre of buoyancy below the centre of gravity of the ship, and R is the distance of the centre of buoyancy below the metacentre. For R we may write

$$R=\frac{I}{V}=\frac{CL^3b}{KLbd}=\frac{C}{K}\frac{L^2}{d},$$

in which C and K are constants like c and k, depending on the form of the ship. From this it appears that the length of the ship plays the part that is taken by the beam in transverse stability.

Interior Carenes.—The geometric considerations which apply to floating bodies apply directly to a volume of liquid contained in a vessel of any form. This is a very important feature in the discussion of the effect of admitting water to the compartment of a ship, and of carrying liquid cargo in tanks or compartments that are not entirely filled, and also of the filling and emptying of such tanks and compartments.

Contact of a Water-line on the Surface of Water-lines.—Each isocarene water-line touches the surface of water-lines at its centre of gravity. For consider two water-lines which make a small angle with each other; they will intersect in a line parallel to the common axis of inclination, and will touch the surfaces of water-lines in two adjacent points. Let one water-line remain fixed and let the other revolve into coincidence with it; as the angle between them approaches zero the point of contact of the moving plane will approach that of the fixed plane, and at the same time their intersection will approach a line through the centre of gravity of the fixed plane; consequently the point of contact of the fixed plane is on a line through its centre of gravity. If another axis of inclination is taken, it can be shown in the same way that the point of contact of the fixed plane is on a line through its centre of gravity and parallel to the new axis of inclination; consequently the point of contact must lie at the intersection of the two lines through the centre of gravity; that is, the centre of gravity is the point of contact of the water-line with the surface of water-lines. In other words, the surface of water-lines is the locus of the centres of gravity of the water-lines.

Curve of Water-lines.—The centres of gravity of the successive water-lines, as a ship is inclined about a given axis, will trace a curve on the surface of water-lines. The projection of this path on the plane of inclinations is called the curve of water-lines.

Projecting Cylinder.—The several water-lines, as a ship is inclined about a given axis, are all parallel to the same axis of inclination and are perpendicular to the plane of inclination. The lines which project the centres of gravity of such water-lines lie in the water-lines themselves, and are perpendicular to the plane of inclination. They are tangent lines since they lie in tangent planes, and they form the elements of a projecting cylinder which is tangent to the surface of water-lines along the path of the centres of gravity of the water-lines. The intersection of this projecting cylinder by the plane of inclination is the curve of water-lines, and the trace of any water-line on the plane of inclination is

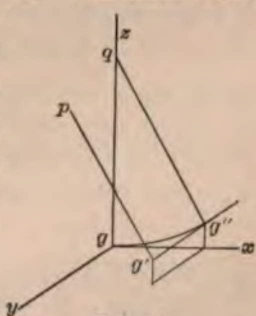


FIG. 76.

tangent to the curve of water-lines. If the floating body is closed, and if it make a complete revolution about a given axis, the path of the centre of gravity of the water-lines is closed, and so also is the projecting cylinder.

Normal to the Curve of Water-lines.—The normal pg', Fig. 76, to the surface of water-lines at the point g' is perpendicular to the water-line of which that point is the centre of gravity. A plane raised through pg' and the projecting line $g'g''$ will be normal to the projecting cylinder, and the intersection $g''q$ of that plane with the plane of inclination will be normal to the section gg'' of the projecting cylinder by the plane of inclination, i.e., qg'' is normal to the curve of water-lines at the point g''.

Coordinates of the Centre of Gravity of Water-lines.—In the discussion of the form of the surface of water-lines Dupin's method will be used, for, though it has at first an artificial appearance, it will be found simple and logical, and will give a convenient parallelism with the discussion of the surface of buoyancy.

Let Fig. 77 represent the intersection of two water-lines which make the small angle $\Delta\theta$, and let Fig 78 represent a partial projection of these water-lines on a horizontal plane. Consider a narrow strip cut from the side of the floating body by vertical planes which are normal to the contour of the water-line WL; $abb'a'$ is the projection of the strip on the horizontal plane, and Fig. 78, A, shows the true form of the section of the side. The strip makes the angle μ with the vertical, which angle is considered to be positive if the side has a flare at the water-line; if the side had a tumble-home, it would be negative. The distance of the strip from the axis gy is x_i, and its height is

$$bc = x_i\Delta\theta,$$

so that the projection on the horizontal plane has the dimension

$$ab = x_i\Delta\theta \,.\, \tan\mu;$$

if the width of the strip along the contour is Δs, the area of the projection $abb'a'$ is

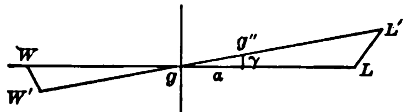
$$x_i\Delta s \,.\, \Delta\theta \,.\, \tan\mu \,.\qquad (27)$$

The area of the corresponding projection on the other side of the axis gy is

$$x_e \Delta s \,.\, \Delta\theta \tan\mu \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

The moment of the original water-line about gy is zero, because g is its centre of gravity. The moment of the projection on the horizontal plane of the new water-line $W'L'$ may be obtained by adding the moment of the strip between L and L' and subtracting

FIG. 77.



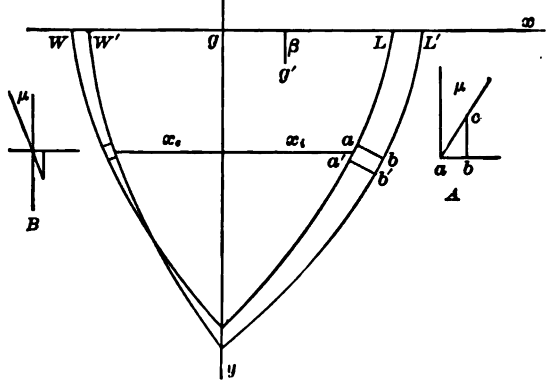


FIG. 78.

that of the strip between W and W'. But when the angle $\Delta\theta$ approaches zero the moment of the new water-line approaches the moment of its projection on the horizontal plane, and the area of the new water-line approaches the area of the original water-line, which may be represented by A.

Let α, β, and γ be the coordinates of the new centre of gravity (g', g'', Figs. 77 and 78) parallel to gx, gy, and gz. Equating the moment of the new water-line about the axis gy to the difference of the moments of the projected strips,

$$A\alpha = \Delta\theta\int x_i(x_i ds \,.\, \tan\mu) - \Delta\theta\int(-x_e)\,(x_e ds \,.\, \tan\mu).$$

$$\therefore A\alpha = \Delta\theta\left(\int x_i^2 ds \tan\mu + \int x_e^2 ds \,.\, \tan\mu\right). \quad . \quad . \quad (29)$$

$$\therefore A\alpha = \mathbf{I}\Delta\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\therefore \quad \alpha = \frac{\mathbf{I}}{A}\Delta\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where **I** is a special moment of inertia obtained by summing up for the entire contour of the original water-line the expression $ds \,.\, \tan\mu$. This moment of inertia is something like the moment of inertia of a line or a fine wire, except that it is customary to consider the wire to have a uniform section and weight per unit of length; but tan μ varies from point to point along the contour and may be positive or negative. Consequently, although the expressions

$$\int x_i^2 ds \,.\, \tan\mu \quad \text{and} \quad \int x_e^2 ds \,.\, \tan\mu$$

are both affected by a positive sign in equation (29) and both contain the square of the distance from the axis, and although ds is always positive, the sum of these expressions may be either positive or negative depending on μ. It can be seen at once that if μ is always positive, that is, if the side has a flare at the water-line all the way round, then **I** is positive; but if the side has a tumble-home for part of the contour of the water-line, then **I** may be positive or it may be negative.

Taking moments about the axis gx and using y for the distance of the projection $abb'a'$ from that axis,

$$A\beta = \Delta\theta\int y(x_i ds \,.\, \tan\mu) - \Delta\theta\int y(x_e ds \,.\, \tan\mu).$$

$$\therefore A\beta = \Delta\theta\left(\int x_i y ds \,.\, \tan\mu - \int x_e y ds \,.\, \tan\mu\right). \quad . \quad . \quad . \quad (32)$$

$$\therefore A\beta = \mathbf{K}\Delta\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\therefore \quad \beta = \frac{\mathbf{K}}{A}\Delta\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where **K** is a moment of deviation with the conditions that have been attached to the moment of inertia **I**.

To get the third coordinate of the centre of gravity of the new water-line after an infinitesimal inclination, the following device is used. Let gg'', Fig. 79, be the curve of water-lines for a finite inclination θ, and $g''s$ the trace on the plane of inclination of the water-line at that angle; $g''s$ is tangent to the curve of water-lines at g''. If the curve of water-lines is the arc of a circle, then $g''s$ is equal to gs and each is somewhat greater than one-half of $g''n$. If θ is not large, we may have for an approximation

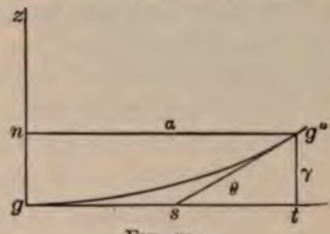


FIG. 79.

$$\gamma = g''t = g''s \sin\theta = \tfrac{1}{2}\alpha \sin\theta;$$

when the inclination becomes infinitesimal we may write

$$\gamma = \frac{1}{2}\alpha\Delta\theta = \frac{1}{2}\frac{\mathbf{I}}{A}\overline{\Delta\theta}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The forms deduced for the coordinates of the centre of gravity of the water-line after a small inclination are the same in form as those deduced for the coordinates of the centre of buoyancy, except that A, the area of the water-line, replaces V, the volume of the carene, and **I** and **K** are the very special if not artificial functions defined on page 146.

Form of the Surface of Water-lines.—In the equation for the vertical coordinate of the centre of gravity of the new water-line,

$$\gamma = \frac{1}{2}\frac{\mathbf{I}}{A}\overline{\Delta\theta}^2,$$

$\Delta\theta$ appears as the square and A is always positive, so that the sign depends on the function **I**, which has been called a moment of inertia; but it has been seen that this function may be either positive or negative, and consequently the coordinate γ may be positive or negative, and the surface of water-lines may be convex or concave at a given point, or it may be partly on one side and partly on the other side of the tangent plane.

If the surface of the floating body is continuous and closed, the surface of water-lines is so also. The surface of water-lines for a ship of the ordinary form is entirely within the body bounded by the skin of the ship and the deck. The surface of water-lines for a catamaran for ordinary inclinations is entirely outside of either hull; for a large inclination it may enter one of the hulls. The surface of water-lines for a circular life-buoy may be partly within and partly without its surface, or it may be wholly outside.

Principal Moments of Inertia.—The properties of the moment of inertia and the moment of deviation of a plane figure which were used in the discussion of the surface of buoyancy cannot be employed directly in the discussion of the surface of water-lines, for the functions **I** and **K**, which have been called by these names, contain the angle μ which the side of the floating body makes with the original water-line, and μ may be either positive or negative, in consequence of which **I** may sometimes be negative. Dupin's method, therefore, does not give a means of determining the sections at all points of the surface of water-lines, but some special cases can be investigated, and these fortunately are the most interesting ones. It is a fact that the investigation of the surface of buoyancy is unnecessarily general, but the investigation of special cases would be little if any easier, and a general investigation of theoretical conditions always has an interest and value.

All ordinary ships are symmetrical transversely, so that when there is no transverse inclination the water-line is symmetrical with regard to a fore-and-aft axis through its centre of gravity, and this symmetry applies also to the inclination of the side of the ship at the water-line. Considering Fig. 78, it appears that if the floating body were symmetrical with regard to a vertical plane

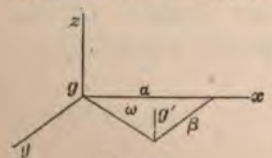


FIG. 80.

through gy, then the element $abb'a'$ at the point (x_i, y) and the element at the corresponding point (x_e, y) would have the same length Δs and the same flare μ, and consequently the expressions

$$x_i y \Delta s \,.\, \tan\mu \text{ and } x_e y \Delta s \,.\, \tan\mu$$

would be equal and have the same sign; the same would be true if

both had a tumble-home. As this would be true for all symmetrical points entirely round the contour, the expressions

$$\int x_i y ds . \tan \mu \quad \text{and} \quad \int x_e y ds . \tan \mu$$

would be equal and **K** would be zero. The fore-and-aft and the transverse axes of such a symmetrical water-line may be called its principal axes, and the moments of inertia of the perimeter affected by the inclination of the side referred to these axes may be called the principal moments of inertia. If there is a flare all the way round the perimeter, the moments of inertia about all axes through the centre of gravity of the water-line are positive, and the principal moments of inertia are the maximum and minimum moments. It is likely that the principal moments of inertia will be the maximum and minimum moments of inertia for any ship when standing erect or when it has any usual change of trim.

Direction of Motion of the Centre of Gravity of Water-lines.—A vertical plane through the centre of gravity g of the original water-line (Fig. 80) and the centre of gravity g' of an isocarene water-line, after a very small inclination, makes an angle ω with the plane of inclination given by the equation

$$\tan \omega = \frac{\beta}{\alpha} = \frac{\mathbf{K}}{A} \Delta\theta \div \frac{\mathbf{I}}{A} \Delta\theta = \frac{\mathbf{K}}{\mathbf{I}}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The centres of gravity of successive water-lines, as the ship is inclined to a finite angle about a given axis, trace a path on the surface of water-lines which, in general, is a curve of double curvature. This is evident from the following considerations: The location of the centre of gravity of a water-line after a finite inclination depends on the form of the surface of the ship at that water-line; and the direction of a vertical plane through the centre of gravity of the original water-line, and the centre of gravity of the new water-line, will depend on the forms of both water-lines, whereas the direction of motion for an infinitesimal inclination depends only on the properties of the original water-line; and since there is no necessary relation between the properties of the original and the new water-lines, the direction of the path of centre of gravity of water-lines is liable to change during a finite inclination. This discussion, which is parallel to that for motion of the centre of buoyancy, applies to any position of any floating body.

If the ship is originally erect, so that the perimeter of the water-line has the kind of symmetry imputed to it in the discussion of principal moments of inertia, then K is zero and the angle ω of Fig. 80 is also zero. This shows that, under the given conditions, the centre of gravity begins to move in a plane of inclination through the centre of gravity of the original water-line, and that plane is tangent to the path of the centres of gravity at the original centre. It is, of course, immediately evident that for changes of trim the centres of gravity of the water-lines remain in the plane of inclination.

Radius of Curvature of the Curve of Water-lines.—The method of obtaining the radius of curvature of the curve of buoyancy can be applied without reservation to the determination of the radius of curvature of the curve of water-lines, giving as a general result

$$\rho=\frac{\mathrm{I}}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Sections of the Surface of Water-lines.—The method for the surface of buoyancy may be applied without reservation, giving, for the radius of curvature of any vertical section through the centre of gravity of the original water-line and the centre of gravity of a new water-line after an infinitesimal inclination about any axis,

$$\rho_1=\frac{1}{A}\frac{\mathrm{I}^2+\mathrm{K}^2}{\mathrm{I}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$\therefore \rho_1=\rho+\frac{1}{\rho}\frac{\mathrm{K}^2}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Principal Sections of the Surface of Water-lines.—If the ship is erect in the original position so that the water-line has the kind of symmetry imputed to it in the discussion of the principal moments of inertia, then K becomes zero and equation (39) reduces to

$$\rho_1=\rho=\frac{\mathrm{I}}{\mathrm{A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The following conclusions have a certain parallelism with those drawn for the surface of water-lines; it will be noted that there are but three and that they are more limited.

1. When a ship is erect the principal sections of the surface of water-lines are made by fore-and-aft and transverse planes.

2. When a ship, originally erect, is inclined about a fore-and-aft axis, the centre of gravity of the water-line begins to move in a transverse plane.

3. Under this condition the radius of curvature of the section of the surface of water-lines by a transverse plane through the centre of gravity of the original water-line is equal to the radius of curvature of the curve of water-lines.

Curve of Water-lines.—The curve of water-lines, being the envelope of a set of isocarene water-lines, is readily drawn.

If the angle between the skin of the ship and the water-line is constant, equation (37), which depends on equation (29), may be written

$$\rho=\frac{\tan\mu}{A}\left(\int x_i^2ds+\int x_e^2ds\right), \quad . \quad . \quad . \quad . \quad (41)$$

in which the parenthesis represents the ordinary moment of inertia of the contour of the water-line.

In the case of a horizontal cylinder terminated by right sections the value of μ for any transverse inclination is constant at each side and is zero at each end. The value of ρ then becomes

$$\rho=\frac{1}{A}\left(\tan\mu\frac{b^2}{4}\int dL+\tan\mu'\frac{b^2}{4}\int dL\right)=\frac{1}{2}\left(\frac{b}{2}\tan\mu+\frac{b}{2}\tan\mu'\right), \quad (42)$$

where b is the half-breadth of the water-line and L is the length, while μ and μ' are the angles at the sides, as shown in Fig. 81; it is to be noted that

$$A=Lb.$$

Equation (42) leads to the following construction of the centre of curvature of the curve of water-lines for a right-ended cylinder: In Fig. 81 draw normals to the contour of the transverse section at the water-lines ad and cf, and bisect the distance between f and d where these normals intersect a vertical through the middle of the water-line; then n, the point of bisection, is the centre of the curve of water-lines.

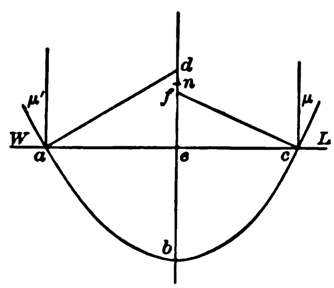


FIG. 81.

If the transverse section is symmetrical, as in Fig. 82, then equation (42) reduces to

$$r=\frac{b}{2}\tan\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and to locate the centre of curvature of the curve of water-lines it is sufficient to draw a normal to the contour of the transverse section at one end of the water-line, and note the point at which it intersects the vertical through the middle of the water-line.

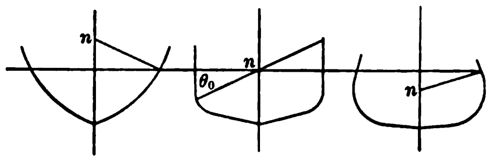


FIG. 82.

If the side of the cylinder has a flare at the water-line, the centre of curvature of the curve of water-lines is above the water-line; if there is a tumble-home, the centre is below the water-line; if the sides are vertical, the curve of water-lines reduces to a point at the middle of the water-line. Although these conclusions apply only to right-ended horizontal cylinders, the ideas aid in forming conceptions of the behavior of actual ships.

CHAPTER V.

ADDING AND MOVING WEIGHTS.

In dealing with the addition and movement of weights, it is convenient to assume that the ship is at first erect and has the normal trim, so that the known properties of the ship may be used as far as possible in solving the problems that arise. It is also convenient to assume that any weight added is at first so placed that it will produce only an increase of draught without changing the trim or causing an inclination; afterwards the weight may be assumed to be moved to its proper location, and the resultant inclination and change of trim can be determined. Again, any motion of a weight may be resolved into vertical, longitudinal, and transverse components; because a vertical motion of a weight affects the stability only and will not produce an inclination unless the ship becomes unstable, a longitudinal motion will produce a change of trim only, and a transverse motion will incline the ship with only a small change of trim. The effects due to the vertical and longitudinal components of the motion of a weight may always be determined independently from properties and curves which are habitually determined during the design of the ship. If the movement of the weight produces only a small change of trim, the inclination due to the transverse component may be readily determined from the usual curves of stability, but if there is a large change of trim, the determination of the effect of that component is more troublesome.

Vertical Movement.—If a weight is moved from a position, as g, Fig. 83, to another position, g', on the same vertical line, the centre of gravity of the ship (and its contents) will be raised vertically. If the ship is in equilibrium before the weight is raised, it will be in equilibrium afterwards; but the equilibrium may

become unstable, on account of the raising of the centre of gravity of the ship.

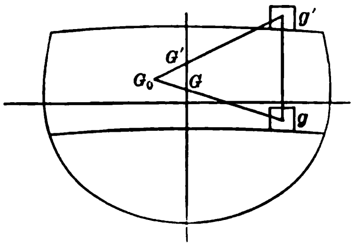


FIG. 83.

If the displacement of the ship is D tons, and the weight of the body moved is w tons, then the ship and its contents can be divided into two portions, w at the point g and $D-w$ at a point G_0, which can be located by drawing the line gG_0 and making

$$G_0G = Gg\frac{w}{D-w},$$

so that

$$D:w::G_0g:G_0G.$$

After the weight is lifted from g to g' the centre of gravity of the ship and its constants will be found at G' on the line G_0g' and dividing it in the proportion

$$D:w::G_0g':G_0G'.$$

Combining the proportions gives

$$G_0g:G_0G::G_0g':G_0G',$$

which shows that the triangles G_0GG' and G_0gg' are similar, and as gg' is vertical, so also is GG'; and finally

$$GG' = \frac{w.gg'}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the distance that the centre of gravity of the ship and its contents has been raised.

If a weight on board a ship is raised vertically, the metacentric height is decreased and the initial stability is diminished. If by such a process the centre of gravity of the ship is raised above the metacentre, the equilibrium becomes unstable, and the ship will take a list to one side.

For example, consider the effect of carrying the boilers on the main deck instead of in the hold, as is the practice on certain lake

steamers. Suppose that the displacement is 4000 tons, and that boilers, and water in them, weigh 250 tons. If the boilers are 15 feet higher on the main deck than they would be in the hold, then the effect of placing them on the deck is to raise the centre of gravity the distance

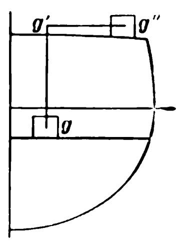
$$GG' = \frac{250 \times 15}{4000} = 0.94 \text{ of a foot.}$$

Movement in a Transverse or in a Longitudinal Plane.—By a process of reasoning like that just given for a vertical movement, it can be shown that a movement of a weight in any transverse plane will not cause the centre of gravity of the ship to leave the transverse plane through its original position; and a like proposition holds for the movement of a weight in any longitudinal plane.

Now a ship is always symmetrical transversely, and a longitudinal inclination or change of trim does not give rise to a transverse inclination. We may conclude that a fore-and-aft movement of a weight will not cause a transverse inclination.

On the contrary, since ships are not symmetrical fore and aft, a transverse inclination is usually accompanied by a change of trim. That change of trim is usually small, and will be neglected in dealing with movements of weights on board a ship.

The movement of a weight in a transverse plane or a longitudinal plane can be resolved into a vertical movement which raises (or lowers) the centre of gravity of a ship, and a horizontal movement which produces an inclination. Thus, in Fig. 84, the movement of a weight w from g to g'' may be resolved into the components gg' and $g'g''$. The first will raise the centre of gravity of the ship by the amount



$$gg' \cdot \frac{w}{D},$$

and the height of the centre of gravity of the ship above the centre of buoyancy will be

$$a' = a + gg'\frac{w}{D}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

FIG. 84.

The horizontal component $g'g''$ will give rise to an inclining couple

$$w \cdot g'g''.$$

Small inclinations, whether transverse or longitudinal, can be calculated by the metacentric method, using the corrected value a' in the metacentric heights $r-a'$ and $R-a'$. The equations are:

$$\text{Transverse} \ldots\ldots\ldots\ldots w \,.\, g'g'' = D(r-a')\theta$$
$$\text{Longitudinal} \ldots\ldots\ldots w \,.\, g'g'' = D(R-a')\theta$$

Transverse inclinations are conveniently expressed in degrees as follows:

$$\text{angle of inclination} = \frac{180}{\pi}\theta = \frac{180}{\pi}\frac{w \,.\, g'g''}{D(r-a')}. \quad . \quad . \quad (3)$$

Longitudinal inclinations are usually calculated from the moment to change trim one inch, which is one of the properties of the ship habitually determined and recorded together with the tons per inch of immersion, etc., as explained on page 81.

For example, suppose that a certain ship when loaded is found to be trimmed by the stern and that it is to be brought to a proper trim by shifting water-ballast from the after trimming-tank to the forward trimming-tank. The displacement of the ship is 7500 tons, the length between perpendiculars is 360 feet, the moment to change trim is 800 foot-tons, and the distance between the trimming-tanks is 330 feet. If 50 tons of water-ballast is shifted from the after trimming-tank to the forward tank, the moment to change trim is

$$330 \times 50 = 16500 \text{ foot-tons,}$$

and the change of trim is

$$16500 \div 800 = 20.7 \text{ inches} = 1 \text{ foot } 8.7 \text{ inches.}$$

Inclining Experiments.—The position of the centre of gravity of a ship may be located approximately by calculation from the weights and location of the members of the structure, the engine and boiler, the fixtures and cargo, or other burdens. But such a calculation is always incomplete, and is unsatisfactory unless checked by some direct experimental determination of the position of the centre of gravity. When an experimental determination of the centre of

gravity of a ship has been made in one condition of loading, the effects of various methods of loading or of the addition or subtraction of weights in general can readily be allowed for.

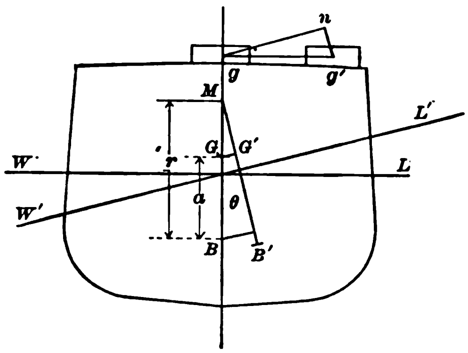


FIG. 85.

The following brief description will show the method of making inclining experiments to determine the location of the centre of gravity of a ship:

Let a weight w be moved from amidships to one side from g to g' (Fig. 85), thus producing a small inclination. The inclining moment will be very nearly

$$w \cdot gg'.$$

The righting moment will be very nearly

$$D(r-a)\theta,$$

where D is the displacement of the ship including the weight w, $r-a$ is the metacentric height, and θ is the angle of inclination in angular measure. Consequently

$$w \cdot gg' = D(r-a)\theta.$$

$$\therefore r-a = \frac{w \cdot gg'}{D\theta}$$

But the value of r may be known from the lines of the ship, as well as the location of the centre of buoyancy, so that the value of a and the location of the centre of gravity can be readily found after an inclining experiment has been made.

Usually two or three weights conveniently placed on the upper

deck are employed, instead of only one weight. For this purpose pig iron on trucks is very appropriate. The angle is measured by at least two plumb-bobs, one well forward and one aft. These plumb-bobs should be suspended on strings that are at least ten feet long and should move over divided scales. The distance the line moves over the horizontal divided scale divided by the distance of the scale from the point of suspension of the plumb-bob gives the tangent of the angle of inclination; if desired, the scale can be divided so as to give the inclination in degrees. Some observers prefer to allow the plumb-bob to swing over a plain batten on which the mid-position and the deviations from that position when the ship is inclined can be marked directly and afterwards measured with a foot-rule.

The plumb-bobs are sometimes hung down hatchways, or they may be suspended from standards on deck, so that all observers will be under the eye of the one responsible for the test, who can at the same time personally verify the location of the weights. On a large ship one responsible observer can pass from place to place without vitiating results. To avoid swinging of the plumb-bobs in the wind they may be boxed in, leaving only a place for the observation of the motion of the line of the plumb-bob over the divided horizontal scale.

The following precautions must be taken to insure satisfactory results:

1. The ship must be in quiet water; a basin or a dock is to be preferred.

2. There should be but little wind, and the ship should be placed with the head (or stern) to the wind. This condition is often unavoidably violated.

3. All lines from the ship to the shore should be cast off or made slack. A head or a stern line to prevent drifting may be allowed.

4. The hold should be pumped dry, and all loose objects and materials that are liable to shift should be removed or secured; the observers and the laborers for moving the weights must be in appointed positions when observations are taken. Persons not needed for the test should be sent ashore. Failure to follow this condition scrupulously is more liable to give unsatisfactory results than any other circumstance.

5. The draught of the ship must be measured at the bow and stern, to afford data for the calculations of the locations of the centre of buoyancy and the metacentre.

6. A systematic record must be made of the condition of the ship, and the amount and location of the movable weights.

To begin a test the weights are placed amidships and the ship is brought into an upright position. The weights are then moved to one side, say to the starboard, and observations are taken. The weights are then returned amidships, and observations are made to see if the ship is erect; if not, the deviation is noted. The weights are then moved to the opposite (port) side, and new observations are taken. As it is somewhat difficult to bring the ship to an erect position, some observers prefer to omit observations at the mid-position and take observations at the extreme inclinations only. The work may be repeated to check the results. The weights should be shifted smoothly and regularly, and the inclination, after the weights are moved, should be watched for a little while. Any unexpected or irregular movement, and especially any increase of inclination after the weights are shifted completely over, should be carefully inquired into, as such a movement is probably due to shifting of some object or material; the presence of water in the hold or double bottom is to be suspected even though ordinary precautions have been taken to pump out the bilges.

Some observers get a series of inclinations with increasing angles by moving the weights from the middle to a series of stations, the greatest inclination being, of course, obtained by moving the weights to the greatest possible distance. From such a series of inclinations there may be calculated a series of metacentric heights, and the most probable value may be inferred from inspection or from a diagram like Fig. 86, where the inclinations are laid off for abscissæ and the metacentric heights for ordinates. The construction of such a diagram has the further advantage that the effect of a variation of the wind during the experiment may be detected and allowed for. Fig. 86 is drawn with the assumption that the inclinations are symmetrical on the two sides; if here is at

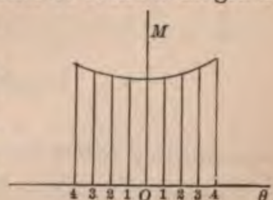


Fig. 86.

steady breeze the figure may be unsymmetrical, and if there is a change in the breeze the figure may be discontinuous. The inequalities of the metacentric height have been purposely exaggerated in the figure. It has been suggested that the true metacentric height is given by the ordinate at zero; but considering that all the inclinations are small and that the least error of observation is to be attributed to the greatest inclination, it appears probable that the best result is to be had from that observation. If the figure is unsymmetrical or discontinuous, the observer may use his discretion in distributing the errors or inequalities so as to get a new symmetrical diagram from which the metacentric height at the maximum inclination may be inferred.

An inclination of at least three degrees from the vertical should be obtained and the weights should be proportioned to give such an inclination; a larger inclination can seldom be obtained on a large ship.

As an illustration of this method, suppose a ship having a displacement of 4200 tons to be heeled by moving a weight of 50 tons from amidships to the side, a distance of 20 feet. Suppose that the inclination is shown by a 15-foot plumb-bob, the foot of which swings off 16.7 inches.

Referring to Fig. 85,

$$GG' = \frac{w \times gg' \cos BMB'}{D}$$

and

$$GM = \frac{GG'}{\sin BMB'},$$

so that

$$GM = \frac{w \times gg'}{D \times \tan BMB'}.$$

Now

$$\tan BMB' = \frac{16.7}{15 \times 12}.$$

Substituting values given above in equation for GM,

$$GM = \frac{50 \times 20 \times 15 \times 12}{4200 \times 16.7},$$

$$GM = 2.555 \text{ feet} = r_0 - a.$$

If now the draught is 18 feet, $r=8.75$ feet, and the centre of buoyancy is 11.6 ft. above the keel, the centre of gravity above the keel is

$$11.6+8.75-2.555=17.795 \text{ ft.},$$

or is

$$18-17.795=.205 \text{ ft. below the load water-line.}$$

Increased Immersion.—If a ship, whether erect or inclined, has a weight placed on board in such a position as to give increased immersion only, there will be added to the displacement a layer which is bounded by the original water-line, a new parallel water-line, and the skin of the ship. In order that there may not be any inclination produced by adding the weight, it must be placed over the centre of figure of this added layer. The added weight and the added layer then form a system which is in equilibrium by itself and which will not produce an inclination when added to the ship.

If the added weight and the added layer of immersion form a system which is in stable equilibrium, then the stability of the ship will be increased by adding the weight. If they are in unstable equilibrium, the stability of the ship will be decreased, and may become unstable. Just as for any floating body the weight must be below the metacentre of the added layer, if the weight and layer are to be in stable equilibrium.

If the added weight is small, so that the added layer is thin, as in Fig. 87, the curve of water-lines of the original carene can be used in place of the curve of buoyancy of the added layer, and the centre of curvature C of the cu ve of water-lines may be used instead of its metacentre

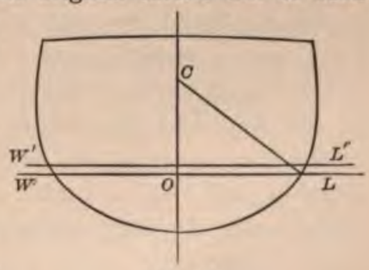


FIG. 87.

Now for a square-ended cylinder the centre of curvature C can be located by drawing a normal to the skin of the ship at the water-line as indicated on Fig. 87. Though this method does not properly apply to a ship with a varying form at the water-line, it may be considered that the tumble-home or the flare at the water-line is seldom

large, and that a normal to the skin of the ship at any transverse section will cut the vertical only a short distance above or below the water-line. Consequently there will not be a large error in assuming the metacentre of the added layer to be at the water-line. The conclusion is that the addition of a weight below the water-line usually increases the stability of the ship, and the addition of a weight above the water-line usually decreases the stability.

Position of Weight to give Increased Immersion only.— Most commonly the ship will be erect before the weight is added, and in such case the increase in draught may be determined from the curve of tons per inch of immersion, and the new water-line can be at once located and drawn on the lines of the ship. Should the added weight be small compared with the displacement of the ship, then the thickness of the layer of added immersion can be determined directly from its volume by dividing by the area of the original water-line.

If the added layer is very thin, its centre of figure may be assumed to be at the centre of gravity of the original water-line. If the added layer is not thick, its centre of figure may be assumed to be at the middle of a line joining the centre of gravity of the original water-line to the centre of gravity of the new water-line; in practice this method will usually be sufficient. If the added layer should be too thick to be treated by the method just given, its centre of figure must be determined by the method used for finding the centre of buoyancy of a ship.

Having found the centre of figure of the added layer of displacement, pass a vertical plane through it and the centre of buoyancy of the ship; this plane will evidently contain also the centre of gravity of the ship and the centre of the added weight. Fig. 88 is intended to represent this plane; B is the centre of buoyancy of the ship, and b is the centre of figure of the added layer, while g is the centre of gravity of the added weight, and G is the centre of gravity of the ship. If the ship is erect before the weight is added, this plane will be the fore-and-aft plane of symmetry of the ship; that is, the added weight will be directly over the keel. The centre of gravity of a ship after the weight is added will be at G_x on a line joining g and G, and will divide that line into segments which

are inversely proportional to the weight added and the original displacement of the ship. In like manner the new centre of buoyancy B_x will be on the line Bb, and will divide it into the same ratio, and B_x and G_x will be on the same vertical, as is necessary for equilibrium after the weight is added. The longitudinal position of the new vertical B_xG_x can readily be determined; but since bg is seldom far from BG, the displacement of B_xG_x from BG can commonly be ignored. The vertical positions of G_x and B_x (or of their projections G_x' and B_x') are required for the completion of the problem. Projecting g at g' and b at b', we have

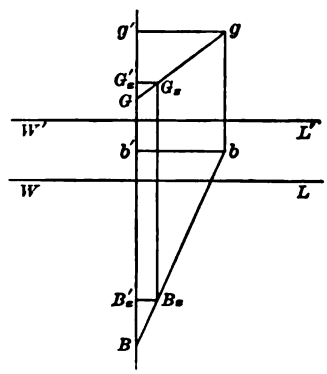


FIG. 88.

$$GG_x' = g'G\frac{w}{D+w},$$

$$BB_x' = b'B\frac{w}{D+w},$$

in which w is the weight added and D is the displacement of the ship, both in tons.

Position of Equilibrium.—After the position at which a certain weight may be added to a ship to give increased draught only has been determined, the next procedure is to find the longitudinal and transverse inclinations produced by moving it to its proper location. If the weight is relatively small, the change of trim and the transverse inclination can be determined by the metacentric method; if the weight is large, it may be necessary to use special methods, which will be explained briefly, leaving the details, which are liable to vary with the problem, to be worked out for each special case.

Change of Trim.—If the change of trim due to the movement or the addition of a weight is large, and especially if the bow or stern becomes immersed, the metacentric method is inapplicable and special calculations must be made either by Barnes' method or by the

method of cross-curves; the procedure for the latter method will be outlined here both because it can be more briefly stated and because it has a wider application. Let Fig. 89 represent the half body plane of the U. S. Light-ship transferred from Fig. 26, page 35, with WL for the load water-line after a weight has been added so as to give increased draught only. After the weight has been given its longitudinal motion (astern in this case) the new inclined water-line will cut the several stations, as indicated by Fig. 88, in a series of uniformly spaced horizontal lines.

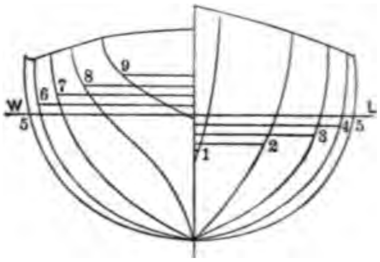


FIG. 89.

The most ready way of determining a curve of longitudinal stability is to draw the curve of buoyancy, and for this purpose it is necessary to determine both the longitudinal and the vertical movement of the centre of buoyancy for each inclination. Bonjean's curves will be found convenient for this purpose, and should be constructed if they are not already drawn.

Taking first the fore-and-aft location of the centre of buoyancy, we have from Bonjean's curves the area at each station up to the inclined water-line, from which we readily compute the displacement up to that plane; and further, by multiplying the area at each station by its distance from the midship section and summing up by the trapezoidal rule we get the moment of the carene with reference to the midship section, and consequently the distance of the centre of buoyancy forward (or aft) of that section. Let this computation be made for a series of water-lines having the same inclination but different displacements. In Fig. 90 lay off abscissæ to represent the displacements cut off by the several inclined water-lines and ordinates to represent the distance of the centre of buoyancy forward of the midship section. Inclinations down at the stern will give locations for

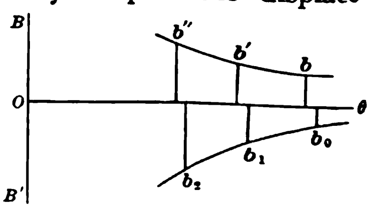


FIG. 90.

the centre of buoyancy abaft the midship section which are laid off below the axis. Each inclination will give a curve like bb'' in Fig. 90, and an assembly of such curves will form a set of cross-curves from which the location of the centre of buoyancy forward (or aft) of the midship section can be at once determined for any inclination and any displacement. If, then, an ordinate is drawn for the proper displacement, it will cross all the curves and will give the longitudinal locations of the centre of buoyancy for that displacement and for all angles.

Bonjean's curves give also the moment of each section about an axis at the top of the keel, so that we may readily compute the moment of the carene cut off by a given inclined water-line with reference to the top of the keel, and also the distance of the centre of buoyancy of that carene above the centre of the keel. Carrying on this computation for several water-lines having the same inclination, we get the material for a new cross-curve, like $bb'\ b''$, Fig. 91, in which the abscissæ are displacements and the ordinates are distances of the centre of buoyancy above the top of the keel. Another curve differing slightly from bb'' will represent the vertical location of the centre of buoyancy for inclination down at the stern. It is omitted from the diagram to avoid confusion. A series of cross-curves will be drawn for several inclinations as in the determination of the longitudinal location of the centre of buoyancy. In practice the two diagrams (Figs. 90 and 91) will be combined, and then one ordinate at the proper displacement will give at once the longitudinal and vertical location of the centre of buoyancy for any inclination.

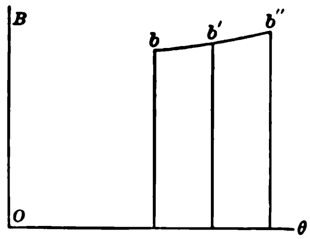


FIG. 91.

From the two series of cross-curves (Figs. 90 and 91) we may readily find the coordinates of the centre of buoyancy for the proper displacement after the weight is added and for several angles of inclination, and consequently may draw the longitudinal curve of buoyancy $B'B_0B_1$ in Fig. 92.

To find the righting arm for the ship (at the proper displacement

including the added weight) for any inclination, it will be sufficient to draw the new vertical BZ', Fig. 92, through the corresponding location B of the centre of buoyancy, and drop on it the perpendicular Gg, which latter is the required righting arm.

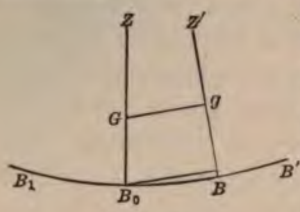


FIG. 92.

The longitudinal curve of buoyancy is very flat, consequently a very few points will be sufficient to locate it, provided they are well distributed. It is probable that the arc $B'B_0$ may be replaced by an arc of a circle, in which case it will be sufficient to determine the extreme point only; the after-branch B_0B_1 will probably have a different radius, and at any rate a point on it should be determined also. It may be that the arc $B'B_1$ can be treated in some cases as a horizontal line; in that case it will be sufficient to determine the curves of Fig. 90 and omit those of Fig. 91.

Since the bow and stern of a ship are not alike, the curve of statical stability for longitudinal inclinations will have two dissimilar branches, that which represents righting moments (or arms) for a trim by the stern being the steeper for most ships.

Thus far it has been assumed that the method of cross-curves will be chosen, and that method probably is most convenient for a complete determination of the curve of buoyancy and the curve of stability. If, however, it is considered that the curve of buoyancy can be replaced by the arc of a circle, and if, consequently, the determination of one point of the curve is sufficient, then Barnes' method may be preferred. As with the application of this method to transverse inclinations, we may begin by drawing the inclined water-line through the middle point of the original water-line, and compute the volume and moments of the immersed and emerged wedges; if the volumes of the wedges are unequal, a corrective layer must be computed in the usual way. Bonjean's curves may be conveniently used for this purpose, since they give the areas of any section up to the original and to the inclined water-lines, and consequently we can get the area of that section between the two water-lines.

If the longitudinal inclination is large enough to submerge the

deck at one end, then the transverse sections at the stations near that end will be measured up to the highest deck that will exclude water at the given inclination. The upper or weather deck will ordinarily be chosen for this purpose; it will be wise, as a rule, not to take account of a forecastle or poop-deck, if such a deck is found at the immersed end of the ship, but this question must be decided for each case.

Having the curves of longitudinal stability, the position of the ship after the longitudinal movement of the added weight may be found by aid of a curve of inclining moments *man*, Fig. 93, laid

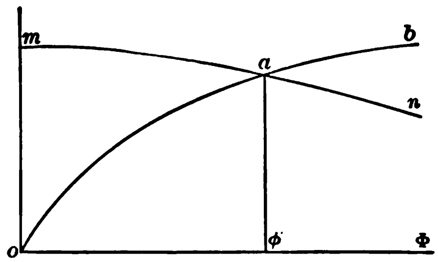


FIG. 93.

off with values of $wl \cos \theta$ for ordinates, in which w is the weight added and l is the distance forward (or aft) of the new centre of buoyancy after the weight is added. The position of equilibrium will be ϕ, where the curve of inclining moments cuts the curve of righting moments.

Transverse Inclinations.—Two cases may be distinguished in dealing with the transverse inclination due to the transverse component of the motion of the added weight from the position where it produces added displacement only to the position required by the problem, depending on whether the change of trim is small or is large.

If the change of trim is small, its effect on transverse stability will also be small, and we can make use of a curve of transverse stability which can be determined in the usual way from the lines of the ship in the erect position, but with the displacement which the ship has after the weight is added.

Suppose that *Oabcd*, Fig. 94, is such a curve of transverse stability, and that the curve *mac* represents the curve of inclining moments due to the transverse component of the displacement of the added weight, calculated by the expression $Wb \cos \theta$, where b is the transverse component of the displacement of the weight, and θ is the transverse inclination. The position of equilibrium is at θ', where this curve cuts the curve of stability.

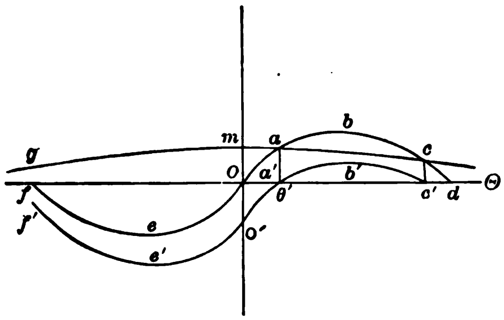


FIG. 94.

The stability after the ship is inclined to the angle θ' is represented by the curve $O'a'b'c'$, constructed by subtracting ordinates of the curve *mac* from corresponding ordinates of the curves of statical stability. It is convenient to draw the other branch of the curve *Oabcd* in the third quadrant as represented by *Oe*, and then the curve of stability after inclination can be continued as shown at $O'e'$, which shows increased stability for inclinations of the ship away from the side toward which the weight has been moved.

Thus far it has been tacitly assumed that the addition of a weight so as to give increased draught only will not give rise to instability. It is very unlikely that the ship will become unstable longitudinally, but it is very liable to become unstable transversely on account of the addition of a weight to give increased draught. In such case the curve of stability (Fig. 94) will not pass through *O*, but will cross the vertical axis below that point; it may, however, rise above the axis $O\Theta$ and give an intersection with the curve of inclining moments *mc* which shows a position of equilibrium after inclination to a considerable angle.

If the change of trim is large, as is likely to be the case when large compartments are broken open to the sea, then a new curve of transverse stability must be constructed for the carenes which are bounded by inclined isocarene water-lines like $W'OL'$, Fig. 95. The method of cross-curves may be used for this purpose, employing the ordinary lines of a ship, measuring the area at each station up to the intersection of that station with the inclined water-line. The arrangement for integration with a transverse inclination is shown by Fig. 95, with the new vertical line OZ' drawn through O, as in

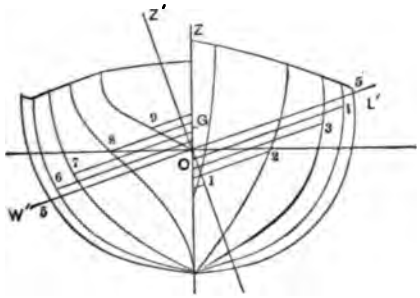


FIG. 95.

the usual conventional manner; the intersections of the water-line with the sections are shown below $W'L'$ for the bow and above that line for the stern, as is proper when a single body plan is used for calculating stability. The usual method with a double body plan is preferable in practice, but Fig. 95 may be more suggestive as an illustration.

The stability can be best treated by the method of cross-curves, using an integrator, which will be adjusted with its axis in coincidence with the new vertical, OZ', Fig. 95, and areas and moments at each station will be measured in the usual way, the section at the first station being measured to 1, at the second to 2, at the third to 3, and so forth. The moment and volume of the carene are determined by the usual process, and the quotient obtained by dividing the moment by the volume is the righting arm projected on the midship section, with the centre of gravity at O. But since the longitudinal inclination seldom, if ever, exceeds 10°, the projection can be used instead of the real length of the righting arm. Allow-

ance must be made for the real location of the centre of gravity of the ship after the weight is added; but again, since the longitudinal inclination is never a large angle, we may take the vertical distance of the centre of gravity from O, Fig. 95, that is, OG in that figure, for this purpose.

The position of equilibrium and the stability after transverse inclination can now be determined as for a ship without a change of trim, and the curve of stability can be drawn as in Fig. 94.

Suspended Weight.—The weight of any body that is freely suspended, as by a rope, is applied at the point of attachment. Thus in Fig. 96 the weight of the parcel of cargo g is applied at the point g', provided that it is free to swing in any direction.

If a suspended body is held by a guy-rope so that it cannot swing, then its weight is applied at its centre of gravity, as is the case with any fixed weight on board.

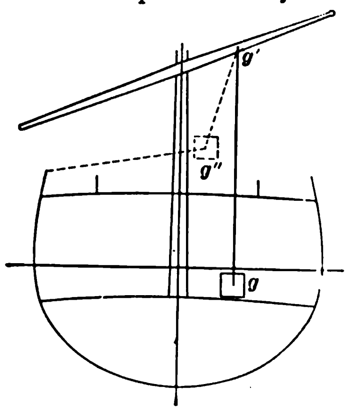


FIG. 96.

A body may be free to swing in one direction but not in another; in this case the weight is treated as applied at the point of suspension for inclinations in the direction in which it can swing, but for inclinations in the other direction it is treated as a fixed body. Thus the transverse stability of a sloop is diminished if the boom is raised from the support by hauling on the topping-lift; but the trim of the sloop cannot be changed by raising the boom out of the support, since the centre of gravity of the boom cannot then move fore and aft any more than it could when supported from the deck.

Conversely, the weight of the body that is poised erect over a given point is applied at that point. For example, a man may stand erect in a light boat that would upset under a fixed weight at a like height.

Movable Weight.—If a body is free to roll (or slide) on a curved path (Fig. 97), its weight is applied at the end of the radius of curvature g' of the path at the point of contact.

The body acts as though it were suspended by a flexible cord wrapped on the evolute of the path of the centre of gravity of the movable body. As a special case we may consider the path of the body to be an arc of a circle, and in that case its weight will be applied at the centre of that circle. This is comparable to a body suspended by a rope.

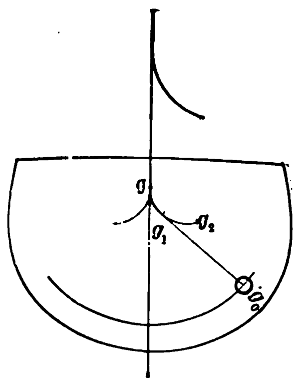


FIG. 97.

A movable body on a convex surface, like a deck, will roll till it is stopped by some obstruction, the bulwarks perhaps. Should the vessel roll, the body is liable to run down the deck violently, from side to side. The tendency of such a rolling body is to check rolling, since it must be raised by the ship at each roll; its energy is expended on the obstacles against which it may strike.

Conversely, it is necessary that a movable body shall be urged up hill against the inclination of the deck in order to set a ship to rolling. To roll a ship artificially, large bodies of men have been set to running across the deck up hill, timing their motions so that they may arrive at a side at the end of the upward roll for that side.

Liquid Cargo.—If a liquid cargo is carried in a closed tank that is kept full, it has the same effect as a homogeneous cargo of the same weight. If the tank is only partly filled, the centre of gravity of the liquid moves from side to side as the ship rolls, and it acts like a suspended or movable body.

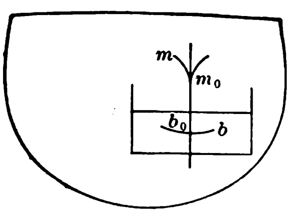


FIG. 98.

Thus, in Fig. 98, the liquid in the tank will have bb_0 for the path of its centre of gravity. This path is the curve of buoyancy for the interior carene, consisting of the liquid in the tank. The evolute of the path b_0b is the metacentric m_0m of the interior carene. These curves are found by the same methods as are employed for drawing the metacentric and curve of buoyancy for an exterior carene.

For small inclinations the metacentric method can be applied to interior carenes, that is, to movable liquid cargoes, just as it can to exterior carenes. Looking at the problem in this way, it appears that the effect of fluidity of the liquid in the tank (Fig. 98) is to raise the point of application of its weight from the centre of buoyancy b_0 to the metacentre m_0.

Water-tight Compartments.—To protect their buoyancy and stability, the hulls of iron and steel ships are divided into a number of separate compartments by transverse bulkheads. Sometimes a longitudinal bulkhead over the keel gives further subdivision.

To give adequate protection to the buoyancy, that is, to prevent the ship from sinking when the skin is broken open in a collision, the compartments should be so numerous that the flooding of two adjacent transverse compartments shall not use up the reserve buoyancy of the ship. The inconvenience of loading and unloading from numerous small compartments tends to limit the number of bulkheads in merchant-ships, so that often not more than one compartment can be flooded without sinking the ship. In case such a ship is injured at or near a bulkhead, there will be great danger of immediate loss of the ship. With the engines and boilers in the middle of the ship there will be at least five bulkheads, namely, a collision-bulkhead near the bow, a bulkhead between the fore-hold and the boilers, a bulkhead separating the engine-room and boiler-room (which, however, may not be water-tight and has a passage from one room to the other), a bulkhead between the engine-room and the aft-hold, and finally a bulkhead near the stern. The door from the engine-room to the boiler-room is intended to be water-tight, but is usually open and cannot always be shut after an accident. To provide against sinking, when one compartment is flooded, there should be at least one bulkhead in each hold. Large passenger-ships have more numerous compartments, and frequently are so designed as to guard against sinking, provided that the doors which must be allowed for convenience of working the ship and for use of the passengers can be closed quickly when required. All water-tight doors in bulkheads between compartments should be provided with gear to close them from some of the upper decks.

All warships, except the smallest, and many passenger-ships

have twin screws and duplicate engines and boilers separated by a longitudinal bulkhead. This longitudinal bulkhead is intended to avoid disabling both sets of machinery in consequence of such damage as will admit water to one engine-room or one boiler-room. But if even one compartment on one side of a longitudinal bulkhead is flooded, the ship will take a dangerous list. Should two adjacent compartments on one side of a longitudinal bulkhead be flooded (as by an injury near the transverse bulkhead between them), it is almost certain that the ship would capsize; this is true of both merchant- and war-ships. It is, therefore, a question whether there is any gain in safety from the presence of a longitudinal bulkhead. When a ship has a longitudinal bulkhead, it is probably a good way to keep doors open to give communication between the pairs of transverse compartments. Then if the engine-rooms, for example, are flooded, it may be possible to isolate one engine-room by closing the door between the engine-rooms, and pump out the other; at the same time a boiler-room or other compartment on the other side of the ship must be filled to keep the ship erect. If the two compartments separated by a longitudinal bulkhead are filled with water to the same height, the ship will of course remain erect, and will suffer much less loss of stability from the mobility of the water than if there were only one compartment. This is readily shown to be true for a rectangular transverse section like that in Fig. 99.

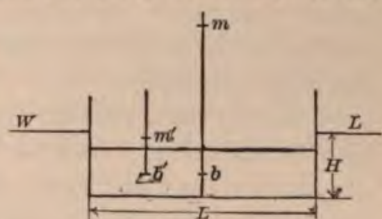


FIG. 99.

The distance of the metacentre above the centre of buoyancy of the interior carene without the longitudinal bulkhead will be

$$r=\frac{i}{V}=\frac{\frac{1}{12}LB^3}{LBH}=\frac{1}{12}\frac{B^2}{H},$$

where L, B, and H are the length, breadth, and depth of water for the compartment. With a longitudinal bulkhead the breadth of a compartment becomes $\frac{1}{2}B$, and the distance of the metacentre above the centre of buoyancy is

$$r'=\frac{i'}{V'}=\frac{\frac{1}{12}L(\frac{1}{2}B)_3}{L(\frac{1}{2}B)H}=\frac{1}{4\times 12}\frac{B^2}{H}=\frac{1}{4}r;$$

that is, the fluidity of the liquid load in the two compartments has only one fourth the effect it would have in a single compartment across the ship.

Floating docks and like structures which are ballasted with water may have their stability increased during the process of filling or emptying ballast-tanks by a number of longitudinal bulkheads in such tanks.

Oil-carrying Steamers.—Ships which are designed to carry a liquid cargo, such as petroleum in bulk, have a very complete system of transverse bulkheads and a longitudinal bulkhead. This is quite as much to limit the dynamic action of the oil when the ship is pitching and rolling in a seaway as to provide against loss of stability, for the compartments are always completely full when the ship is carrying oil. To provide for expansion and contraction of the oil, each compartment has an expansion-trunk large enough to keep the compartment always full, but with so small a free surface that the fluidity of the oil will not cause much loss of stability. The oil is carried entirely beneath the main deck, and the expansion-trunks are commonly placed amidships, extending from the main to the upper deck. In some vessels the trunks form a continuous structure all filled with oil, while in others the trunks are alternated with spare coal-bunkers.

Sometimes the machinery of the ship is placed right aft with the holds forward of it. There is then a coffer-dam two frame-spaces long between the boiler-room and the first oil-compartment, to prevent leakage from the oil-compartment finding access to the boiler-room or coal-bunker. Sometimes the engines are in the middle, in which case there are two holds for oil, one forward and the other aft, and two coffer-dams must be provided, one between the forward hold and the boiler-room, and one between the after-hold and the engine-room. Provision must be made for intercepting leakage from the shaft-tunnel into the engine-room without interfering with access to the tunnel. The coffer-dams may be filled with water or left empty; in the first case leakage can occur only from the coffer-dam into the adjacent oil-compartment, since water is heavier than oil; if the coffer-dams are empty, the weight of water necessary to fill them is avoided, but the leakage of oil or gas into them is

liable to form an explosive mixture. The simplest construction is found in ships which have the machinery aft and only one hold for oil, but a considerable amount of water-ballast is required to trim the ship when empty.

Some oil-carrying ships are designed to have sufficient stability when empty or with just enough water-ballast to give proper trim; this arrangement is likely to give excessive stability and an uneasy motion when the ship is filled. Other ships require to have some of the oil-compartments filled with water when the ship is returning without oil. In either case the load on the ship, without oil, is very unevenly distributed, and the stresses on the structure are apt to be excessive. Account must be taken of these conditions in the design and construction of the ship, and also of the fact that the pressure of the oil on the shell plating tends to open the riveted joints. Attention must be called also to the fact that the pressure of the oil in the compartments is exaggerated by dynamic action when the ship is rolling and pitching.

The filling and emptying of compartments is, of course, done in quiet water, and it is sufficient to be certain that the ship shall not take a dangerous list during that operation. A very considerable list may give rise to no inconvenience, especially when the ship is nearly empty and has plenty of free-board. Some oil-steamers are designed to have all the compartments filled or pumped out at once; they always take a list during that process, which is expected and provided for in the design.

Semi-liquid Cargoes. — Ships often carry grain and similar material in bulk. Such cargoes behave like solid, well-stowed cargoes until the ship rolls to an angle at which the material will slide. Grain begins to slide at about 26° if the inclination is slowly increased, and at a less angle if the inclination is rapid. There is no effect from the partial mobility of grain during loading and unloading, since the ship is then erect. But there is much danger that the cargo may shift when the ship is rolling at sea and produce a list from which the ship will not recover, but which may be increased by subsequent rolling. It is not easy to completely fill the hold with grain, for the grain may settle after the ship has gone to sea. Again, there is danger that the grain may become wet

and swell, so as to burst the deck if the hold is quite full. To minimize the danger of shifting a cargo of grain, shift-boards are put in over the line of the keel and near the surface of the grain, and for further security bags of grain are sometimes piled on top of the loose grain.

Coal-carrying steamers are subjected to a like action, and sometimes they are built with inclined longitudinal bulkheads which cut off a part of the hold at the wings under the beams, to lessen the danger from shifting cargo and to aid in trimming the coal as it is loaded.

Piercing Compartments.—The most important problems to be dealt with in this chapter are those that arise when large compartments of a ship are broken open to the sea. Two distinct cases arise depending on whether, after a compartment is pierced, water can flow freely from the sea to the compartment (and from the compartment to the sea) or whether such a flow cannot take place either because the aperture is closed or the compartment remains entirely filled. Thus if a compartment is bounded at the top by a deck or flat that is below the water-level, and is filled with water, it is evident that no effect will be produced by putting it in communication with the sea by opening a valve or piercing the skin of the ship. Even if the bounding deck or flat is above the water-level, the compartment when once filled may remain full on account of the pressure of the atmosphere on the surface of the water; but such a compartment would not be filled entirely when pierced unless the ship in consequence should become immersed so as to bring the deck below the water-level. It does not appear that the size of the opening through the skin of an immersed compartment will have any effect on the statical condition of the ship; and further, the entire destruction of the bottom of the compartment will not affect that condition, provided that the compartment remains full of water.

The first case, when a compartment, after being placed in communication with the sea, is not subject to a flow to or from the sea, can be dealt with by the methods developed for the addition of a weight. If the compartment is and remains completely full, that weight is a fixed weight; but if the compartment is only partly filled, we have to deal with a mobile weight or a liquid cargo. If a compartment

below an under-water deck or flat is air- or water-tight, then when the skin is pierced water will enter until the pressure in the compartment is equal to that of the sea-water outside, the air in the compartment being compressed to that pressure. Since air rapidly leaks out of a small orifice, it is unsafe to depend on air in a compartment to keep out water unless the supply is continually renewed by an air-compressor; consequently it will be well to treat such a compartment as entirely filled with water.

To find the effect of completely filling an under-water compartment, we will find the weight and centre of gravity of the water that the compartment will hold, allowing for any machinery or fittings in the compartment, but not for fuel, stores, or any material that is liable to be consumed, unless it may be that for some particular case the stores or other material in a given compartment are known. Then proceed to find the position of equilibrium and the stability in that position, treating first the longitudinal inclination and then the transverse inclination, as indicated on pages 164 and 169.

If a compartment not in free communication with the sea is partially filled with water, the point of application of the weight of that water in the erect position will be the metacentre of the interior carene. The longitudinal inclination produced by the liquid load may be found with the assumption that the point of application of that load remains at the metacentre of the interior carene, and the same assumption may be made for moderate transverse inclinations. For large transverse inclinations allowance should be made for the fact that the point of application of the liquid load moves along the interior metacentric curve; it is somewhat troublesome to do this, and it is likely that problems involving such large transverse inclinations will be too indefinite to warrant the labor required.

In dealing with a compartment that is in free communication with the sea, the most convenient way is to assume that such compartment is removed from the hull of the ship, so that we have now a new carene without the displacement of the damaged compartment. If a considerable portion of the volume of the compartment is taken up by an engine, boiler, or other fixture, allowance must be made for it. The volume of the fixture may be replaced by a cylindrical figure having its axis passing horizontally through the centre

of figure of the fixture, and reaching from end to end of the compartment. When the sections of the ship at the compartment are measured to find the volume and moment of the carene, the area of the section of this cylindrical figure will be included in that measurement.

Now find from a special curve of tons per inch of immersion the location of the water-line to give the same displacement as the intact carene, for the injury to the compartment has not altered the weight of the hull and its contents. Next, find the position of the centre of buoyancy of this new carene. It will usually be at a different height from the centre of buoyancy of the intact carene, and will also be forward (or aft) and to one side of that point.

First, let us assume that the centre of gravity is moved to a point vertically over the centre of buoyancy of the new carene, so that the ship may be erect, thus enabling us to use the ordinary lines of the ship for calculating stability, making allowance, of course, for the damaged compartment. The longitudinal inclination will be due to an inclining couple, which has the displacement of the ship for its force and the longitudinal distance between the original and the new centres of buoyancy for its arm. The method on page 164 will allow us to find that inclination. Again, the transverse inclination will be due to a couple having also the displacement for its force and the transverse distance between the centres of buoyancy for its arm. The method on page 168 will enable us to find the position of equilibrium, and the stability of after inclination.

The work outlined for finding the effect of piercing compartments, especially when in free communication with the sea, calls for a large amount of calculation even when an integrator is used, and abbreviated methods are employed. While extreme accuracy is unnecessary in this work, crude approximations are likely to be misleading; the work should be carried through properly or not attempted.

Use of Small Models.—The effect of adding a weight, and especially of flooding compartments, may be conveniently studied by the aid of small models, as advocated by Bertin.* The most exact results will be obtained from a metallic model properly subdivided

* Soc. Naval Archts. and Marine Engs., vol. 2.

by bulkheads and decks, and with allowance for volumes occupied by the engines, boilers, and other large fixtures. Coal, stores, and ammunition can also be allowed for if desirable. The model must, of course, be ballasted so as to float at the proper water-line and to have the proper location of the centre of gravity. Water may then be admitted to compartments as desired, and the position of equilibrium and the stability afterwards can be found by experiment. The cost of such a model will prevent the use of this method for any except important cases, such as the cause of the capsizing and sinking of the British ship *Victoria*.

A wooden model of the ship can be made for a reasonable price, and can be cut so that blocks representing flooded compartments can be removed. The block or blocks removed must be replaced by lead weights which have the same weight and have their centres at the centre of figure of the blocks removed, so that the weight and centre of gravity of the model may remain unchanged.

The model will carry on its deck a transverse bar with a sliding weight for the sake of producing additional inclinations and measuring the inclining moments required to produce those inclinations. The changes of trim are readily measured on the model, and transverse inclinations are measured by small plumb-bobs near the bow and the stern. The model is to be ballasted to give the proper draught and centre of gravity with this inclining apparatus in place.

To find the effect of flooding a compartment, the block representing it is removed and replaced by a lead weight, and the model is set afloat in a tank with the inclining weight in the middle of its bar. The change of trim and the transverse inclination are then measured. The weight is then gradually moved to one side, and the corresponding inclinations and inclining moments are measured. Since the inclining moment of the movable weight is equal to the righting moment of the model, a curve of stability for the ship after the compartment is damaged may be readily constructed.

This curve of stability will be like Fig. 100, crossing the horizontal axis at the angle of equilibrium θ_1. The part of the curve from θ_1 to θ_0 is to be determined by moving the inclining weight away from the damaged side of the ship till it floats erect. When the model reaches the angle of maximum stability θ_2 it will

capsize unless prevented, because the inclining moment will be greater than the righting moment for angles greater than θ_3, when the sliding weight is set so as to give that angle. For a given setting of the sliding weight its inclining moment will be proportional to the cosine of the angle of inclination and may be plotted on the diagram, giving a curve like *mbd*, Fig. 100. The intersection of this curve with the curve of stability *abcd* will determine the position of equilibrium for that setting of the sliding weight. The first intersection represents a position of stable equilibrium to which the model will go when the weight is moved out. If the model is forcibly inclined to θ_4 at the second intersection, the model will be in equilibrium again, though the equilibrium will be unstable. This position of unstable equilibrium can be located very nearly by holding the model with a string at an angle greater than θ_4 and pulling upon the string till the model is balanced.

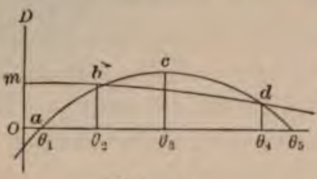


FIG. 100.

The model and the pieces into which it is cut should be varnished or painted, and in preparing for an experiment the surface should be oiled and cracks filled with grease, to exclude water as much as possible.

To avoid trouble from molecular action at the surface of the water, models should be of good size; four to six feet long will be found convenient.

From experiments on models, or from calculations, it will be found that the flooding of a transverse compartment below the water-line will, in general, increase the stability of the ship. If the compartment is near one end of the ship, there may be an inconvenient if not a dangerous change of trim. Flooding compartments above a protective or armored deck always reduces the stability to a dangerous degree, since the moment of inertia of the water-line is much reduced and the metacentric height is correspondingly diminished. Of course a ship is likely to take a dangerous list when a side compartment is flooded, and for this reason longitudinal bulkheads must be considered rather to guard their contents than to add to the safety of the ship. Thus, if only one of a pair of engine-rooms is flooded,

the propulsion power is merely reduced one-half; this, of course, may indirectly save a ship from destruction. As has already been said, the ship must be brought upright again by filling a compartment on the other side of the ship.

Removal of a Weight.—It is evident that the effect of removing a weight is the converse of that of adding a weight; the solution of problems of this nature is so obvious, after the effect of adding a weight is understood, that it is not necessary to state them separately.

Floating Shears, Cranes, and Derricks.—Shears, cranes, and derricks, whether established on quays or piers or mounted on pontoons, are used for placing on board ships such heavy weights as engines, boilers, masts, armor and armament, and for removing them when that is necessary.

The construction and establishment of shears and cranes on quays or piers involve problems of engineering only; the design and construction of floating shears and cranes come properly in the province of the naval architect and involve problems that require the use of the methods of this chapter.

Shears consist of two masts (Fig. 101) which are joined at the head *b* and have their feet *c* separated enough to give lateral stability.

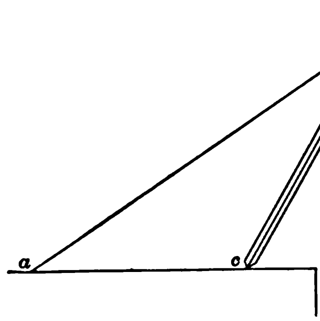


FIG. 101.

The weight *d* is suspended from the head of the shears, and is carried by compression on the masts and tension on the stay *ab*. If the stay *ab* is a rope, the head *b* must not be swung back of the feet *c*; but some shears have a framed stay that can endure compression as well as tension, and then the weight can be swung back through

the legs of the shears. The head of the shears is swung in or out by drawing in or letting out the stay *ab* when that is a rope; if the stay is a rigid framed member, the foot *a* is drawn back and forth in guides. Small shears having wooden masts may be crudely formed for temporary purposes. Large shears are made of steel, the masts being sometimes made of plate, either round or square in section, and sometimes made as lattice struts.

Cranes and derricks have a single arm or boom so supported that it can be swung around a vertical axis, carrying a weight with it. The names are nearly synonymous and are frequently confused. They may be distinguished as follows: The crane has an arm of fixed length and inclination with its head tied to a short post around which the arm can be swung. The derrick has a boom with its head supported from a mast which is held in place by guys. The boom may be swung around and its head may be raised and lowered. But this distinction is not always made.

In the days of wooden sailing-ships, floating shears were commonly used for placing or removing masts. For this purpose a wooden hulk was strengthened near the rail at one side to receive the feet of the shear, and the guy-rope was secured near the rail at the opposite side. The construction of the floating shears was such that the hulk had a list to the side at which the weight was lifted, and this list could be increased by filling a tank with water when the shears were brought alongside the pier to lift a load, thus increasing its reach. When the weight to be lifted was secured to tackle from the head of the shears, the hulk was righted or given a list in the contrary direction by filling a counterbalance-tank, thus lifting the weight from the pier. Before the floating shears were shifted with a load it was customary to attach guy-ropes and draw the load back so that it could not swing, thus increasing the stability. The construction of floating shears is simple and may be crude, but they are inconvenient in use, so that at modern shipyards it is customary to use floating cranes or derricks which have engines for hoisting and swinging the load around. The engine and its boiler commonly swing round with the load and serve as a partial counterweight; there is no counterbalance-tank provided.

The barge or pontoon on which the crane or derrick is mounted

is usually constructed of steel, and may be rectangular in form with the corners rounded. Calculations for stability may therefore be made with facility, since the volumes and moments of the wedges of immersion and emersion can be determined by the ordinary rules of mensuration. Thorough investigations should be made to show that there is sufficient stability under all conditions and a proper freeboard when the pontoon has its greatest draught and greatest inclination, because the stability decreases rapidly when the edge of the deck is immersed.

A floating crane which is used for moving weights to a distance, as from one part of a harbor to another, will have its hull shaped like a barge to facilitate propulsion; it may be towed or may have its own steam-power, and is in fact a form of lighter. Lighters are usually full forward to give good stability when loading and unloading by aid of their cranes or derricks, and are finer aft so that they can be steered. They may be made of steel or wood, and have shallow draught so that they are likely to have sufficient stability.

CHAPTER VI.

GROUNDING AND DOCKING.

A SHIP which touches bottom, and is consequently only partially water-borne, is said to be grounded. Formerly ships were beached or grounded for repairs and painting; now such work is done in docks, except for small craft. The docking of a ship is really a process of grounding, and involves the same problems. When grounded the pressure of the ship against the bottom is equal to the difference between the displacement of the ship when floating freely and the displacement of the carene cut off by the water-line after the ship is aground. It is convenient to consider this force as exerted by the ground on the ship, so that problems may be treated by the methods for the removal of a weight, that is, by the converse of the methods for adding a weight.

Grounding at a Point of the Keel.—Suppose that a ship touches a hard point like a ledge of rock at some point of the keel and that

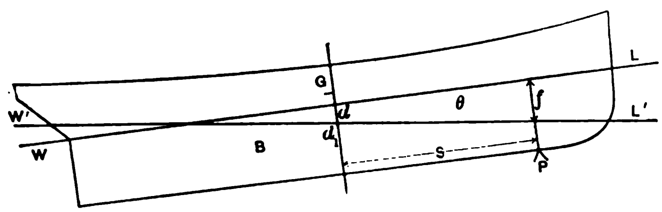


FIG. 102.

the tide then falls; it is required to find the pressure against the bottom and the change of trim.

(1) Assume that the change of trim is not large and that the standard tons per inch of immersion and moment to change trim one inch can be used in the computation. If the length of the ship

(Fig. 102) is L feet, and if θ is the angle of longitudinal inclination after the tide falls, then the change of trim in feet will be

$$c = L\theta \quad \text{and} \quad \theta = \frac{c}{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The fall of the tide f may be measured at P, neglecting the influence of the small angle θ, and the change of draught amidships due to the fall of the tide will be

$$d - d_1 = f - s\theta = f - \frac{sc}{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where s is the distance of P forward of the centre of gravity of the water-line WL. To simplify the problem it is assumed that the centre of gravity of the load water-line is half-way between the draught marks; if this is not approximately true in any case, proper allowance must be made. If the tons per inch of immersion is T, then the loss of displacement and the reaction at the bottom at P is

$$R = 12T(d - d_1) = 12T\left(f - \frac{sc}{L}\right). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let M be the moment to change trim one inch, then the righting moment will be $12Mc$, and this must be equal to the moment of the reaction with regard to the point d, so that

$$sT\left(f - \frac{sc}{L}\right) = Mc. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore c = \frac{sTf}{M + \frac{s^2T}{L}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Having computed the change of trim by equation (5), it may be inserted in equation (3) to find the reaction of the bottom, and also in equation (2) to find the change of draught amidships. Half the change of trim added to the draught amidships will give the draught at the stern, and subtracted from the same quantity will give the draught at the bow.

The values of T and M (tons per inch of immersion and moment to change trim one inch) will be taken for the displacement before grounding to get the first approximation; if the mean draught

after grounding is much changed, it may be necessary to make a second approximation, using T and M for the draught after grounding, as found by the first approximation.

The U. S. S. *Kearsarge* is 368 feet long and has a displacement of 48 tons per inch of immersion and a moment to change trim one inch of 1000 foot tons; if she were to ground on the keel 160 feet forward of the centre of gravity of the water-line and the tide were to fall 6 inches, the change of trim would be

$$c = \frac{sTf}{M + \frac{s^2T}{L}} = \frac{160 \times 48 \times \frac{1}{2}}{1000 + \frac{160^2 \times 48}{368}} = 0.885 \text{ feet}$$

$$= 10.6 \text{ inches.}$$

The decrease of draught amidships will be

$$d - d_1 = \frac{1}{2} - \frac{160 \times 0.885}{368} = 0.115 \text{ ft.} = 1.38 \text{ inches.}$$

The reaction of the bottom will be

$$R = 12T(d - d_1) = 12 \times 48 \times 0.115 = 66.24 \text{ tons.}$$

If the fall of the tide and the change of trim are large, the following method may be followed: Draw an inclined water-line at a convenient angle, as $W'L'$, Fig. 102, and compute the displacement and moment about P of the carene cut off by it, for which purpose Bonjean's curves will be found convenient. If the moment of the inclined carene is different from that of the weight of the ship (concentrated at the centre of gravity) about the same axis, the ship will not be in equilibrium under the conditions chosen, and new conditions must be taken. In the figure the buoyancy at B and the weight of the ship at G are drawn as though they were perpendicular to the original water-line; they will be perpendicular to the inclined water-line $W'L'$, but in dealing with parallel forces and their moments the forces may be turned in any convenient direction, which, in this case, is perpendicular to WL; this arbitrary assumption makes the moment of the weight of the ship about P a constant.

It will be found convenient to draw at the outset a number of water-lines with various draughts at the angle chosen and compute the displacements and moments for the several carenes, and also to measure the draughts at P perpendicular to the inclined water-lines. Now construct a diagram like Fig. 103 with draughts for ordinates, and with displacements and moments for abscissæ, and draw the curves $bb'b''$ for the displacements and $mm'm''$ for the moments. Draw a vertical line gg' to represent the constant moment of the weight of the ship about an axis at P, the point of grounding; its intersection with the curve of moments will give the draught d_0 at which the ship will be in equilibrium at the given angle. Draw also a vertical line aa' to represent the displacement of the ship afloat; the portion a_0b_0 of the abscissa at d_0 will represent the reaction of the bottom. Repeat the computation for a sufficient number of inclinations at convenient intervals and draw diagrams like Fig. 103 for each inclination to determine the draught and reaction at equilibrium for that inclination. From such a set of diagrams a new diagram like Fig. 104 may be drawn which will give the reactions for all draughts, and consequently for all stages of the tide.

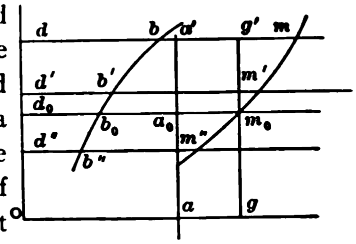


FIG. 103

If the fall of the tide is large and the reaction correspondingly large, there may be danger that the ship will fall over sideways from lack of transverse stability. To investigate this condition for a given draught, assume that a weight equal to the reaction is removed from a point at the bottom of the keel; the virtual centre of gravity will consequently rise; if it passes above the metacentre, the ship will fall over. The method of investigation for this case is substantially the same as that given in the following section.

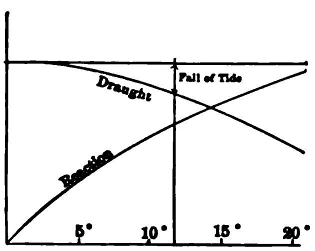


FIG. 104.

Grounding on the Keel.—Let Fig. 105 represent the transverse section through the centre of gravity of a ship grounded along the

keel; for simplicity it will be assumed that the ground is level, and that the ship was on an even keel before grounding.

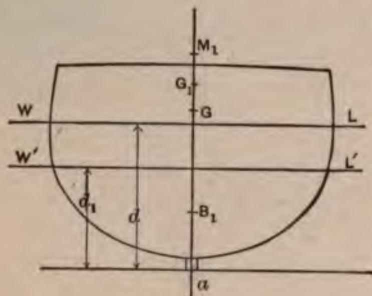


FIG. 105.

Two problems require solution: (1) to find the draught at which the ship will begin to heel over, and (2) to find the angle of equilibrium at all smaller draughts.

(1) This problem may be conveniently solved by the converse of the method used for an added weight. Assume any convenient water-line, as W_1L_1, parallel to the normal water-line WL when the ship is afloat. From the curve of displacements, Fig. 28, page 45, find the displacement up to this assumed water-line, and from the same figure find the height of the corresponding centre of buoyancy B_1 above the keel, and locate B_1 on the diagram. The reaction P of the ground on the keel will be the difference between the displacements to the water-lines WL and W_1L_1.

The effect of the reduction of the draught from d to d_1 may be considered to be equivalent to the removal of a weight equal to P (the reaction) from the point a. The centre of gravity of the ship, after the removal of such a weight from a, would be at G_1, located by the equation

$$aG_1 = \frac{aG \times D}{D-P},$$

in which D is the displacement of the ship when afloat. The height of the metacentre above the centre of buoyancy is given by

$$B_1M. = r_1 = \frac{i_1}{V_1},$$

in which i_1 is the transverse moment of inertia of the water-line W_1L_1, and V_1 is the volume of the carene cut off by the same water-line. If a curve of metacentres of the ship at several water-lines is at hand, the values of B_1M_1 may be obtained by interpolation.

So long as G_1 remains below the corresponding metacentre M_1 the ship will remain erect. When the points coincide the ship becomes

unstable, and when M_1 is below G_1 the ship takes a list. To find the draught at which the ship becomes unstable repeat the calculation for several values of d_1, and plot the distance of G_1 and M_1 from a as ordinates, using the corresponding draughts as abscissæ. The point of intersection of the curves will show the draught at which the two points G_1 and M_1 coincide, and the ship becomes unstable.

(2) To find the angle to which a ship will heel when grounded along the keel, assume a water-line W_1L_1 (Fig. 106), making a given angle θ with the normal water-line WL of the ship afloat, and at a given draught d_1. Find the moment of the inclined carene about an axis through a, and also the moment of the weight of the ship concentrated at G about the same axis; if the two moments are unequal (the usual case), repeat the operation for several water-lines at different draughts, all making the angle θ with WL. Draw a diagram with draughts for abscissæ and moments for ordinates. The moments of the several carenes will be represented by points on a curve, and the moment of the weight at G will be represented by a horizontal line; the intersection of the line and curve will determine the draught at which the ship will be in equilibrium with an inclination of θ degrees. Assume several values for θ and find the corresponding draughts at which the ship will be in equilibrium, and draw a curve with the angles for abscissæ and the draughts as ordinates, as represented by Fig. 107. This curve produced backwards till it intersects the axis at 0° will give the draught at which the ship becomes unstable. It will be seen that the process given under the heading (1) for finding the draught at which the ship becomes unstable is easier and more exact. It is convenient to draw a curve giving the reactions of the ground on the keel in Fig. 107; its construction needs no explanation.

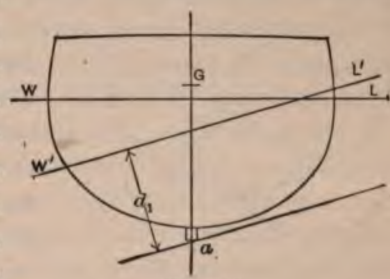


FIG. 106.

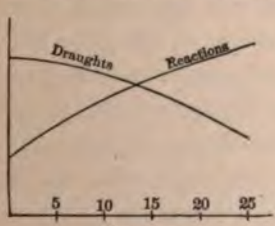


FIG. 107.

The process may be continued till the bilge of the ship touches the ground.

If a set of cross-curves of stability of the ship exists, the labor of this method may be much abbreviated, since such curves give the displacement of the ship at various inclinations and draughts, and also the moments of the corresponding carenes about a fixed axis from which the moment about an axis at the keel can readily be deduced.

Grounding on Keel and Bilge.—If the water continues to fall, a ship grounded on the keel will heel over till the bilge touches the ground also; from that instant the reaction of the ground will be exerted at the keel and at a point on the bilge; at a and at d, Fig. 108, for example. The reaction at d may be found by taking the moment about an axis at a of the displacement D of the ship before grounding cencentrated at G, and subtracting the moment of D', the displacement after grounding concentrated at B', and dividing the remainder by ad.

FIG. 108.

The reaction at a is obtained by subtracting from the original displacement D_1 the inclined displacement D' and the reaction P_d.

When the ship touches at d the reaction P_d is zero; when the reaction at P_a becomes zero the ship is on the point of rolling over farther, so that it will be grounded on the bilge only.

Grounding on the Bilge.—Most merchant-ships which have fairly flat floors will remain grounded on the keel and bilge, or if there is no external keel they will lie on the dead-rise of the floor with a very moderate inclination. Ships with considerable rise of floors and round bilges, like yachts and some war-ships, are likely to roll entirely over on their sides when left stranded as the tide goes out. In such case they are liable to fill either as they roll over or as the tide comes in, and will not float. Yachts which are likely to behave in this way when stranded are sometimes provided with crutches to keep them from falling over.

Docks used for repairing and cleaning and painting ships are of three kinds: (1) dry docks, (2) lifting docks, and (3) floating docks. Marine railways, which will be discussed in another chapter, are used for the same purposes. Ships have been built in docks at English dockyards, where there are more docks than are required for ordinary repairs, and thereby the expense and risk of launching have been avoided.

Dry docks are inclosed basins lined with wood or masonry, and provided with gates to admit vessels for repairs or cleaning.

The methods of providing a proper foundation for a dock, and of constructing the floor and walls to resist the pressure of earth that is often permeated with water, cannot be adequately discussed here. It is sufficient to say that the problems that arise in constructing a large dock may tax the resources of an experienced engineer. The floor of a dock must be able to bear the weight of the largest ship it can admit, concentrated near the keel; the floor must also exclude water and may be subjected to an upward hydrostatic pressure from water that percolates through the earth on which the foundation rests. The side walls are always inclined, and are commonly made in steps for convenience in working in the dock. They are in the condition of retaining walls which are exposed to a semi-fluid pressure.

The gates of a dock were formerly made like the gates of a canal lock; that is, they were in two parts, opening at the middle, and hinged at the outer edges. They were so made that the pressure of external water tended to hold them shut when the dock was pumped out.

Modern docks have the gate in one piece in the form of a deep pontoon that will just fill the entrance to the dock. This pontoon is so shaped and ballasted that it will float erect in water of the least draught at which ships enter the dock, and it is sunk when in place by admitting water. The pontoon must have a sufficient metacentric height when light and when filled with water to insure an erect position.

The ship to be docked, if of considerable draught, commonly enters the dock at or near high tide, and is frequently lightened to reduce its draught, and also to reduce the chance of straining when

it settles on to the blocking. It is desirable, though not imperative, that the ship shall be on an even keel. Lines are carried from each bow and each quarter to the sides of the dock to aid in centring her. The gates are then closed, and the water is pumped out by powerful centrifugal pumps. At the same time blocking is placed under the keel and the bilges, and shores are placed between the sides of the ship and the sides and bottom of the dock.

It has been shown on page 188 that a ship which is grounded along the keel will remain erect till the water has fallen some distance. Advantage is taken of this to make the ship take the blocking under the keel before side shores are placed. It also allows pumping to progress continuously from the time the gates are closed, as there will usually be time enough to place the shores before the ship is in danger of taking a list. Properly the safe draught for the ship when grounded along the keel should be determined before the docking is begun; if that has not been done, the ship must be watched to detect any tendency toward listing, so that the pump may be stopped if necessary. If a ship begins to take a list, it may be necessary to readmit enough water to make it right again.

After the repairing or cleaning and painting is finished, water is admitted through sluices till it attains the external level, whereupon the ship is released and floated out.

Lifting Docks are fixed in place and depend for their stability on some fixed structure or mechanism. There are two kinds of lifting docks; one has a pontoon which is sunk to receive the ship, and pumped out to raise her; the other has a framework or platform for carrying the ship which is raised by hydraulic rams. The latter may have a pontoon to aid in raising the ship.

Fig. 109 represents the first kind of lifting dock, which depends entirely on a pontoon that can be sunk to receive a ship and pumped out to lift her. The pontoon B is rigidly attached to a frame A which is held erect by the linkage $aba'b'$, reaching to a masonry wall or embankment. The frame A may be made open, so that water may pass freely through it, or it can be a closed box girder, which can serve also as a pontoon to aid in lifting the ship, or the bottom part may be closed and the top open. When the pontoon B is submerged, whether loaded with a ship or not, the dock is liable

to be unstable, even though divided into several longitudinal compartments, and then must depend entirely on the linkage $aba'b'$ to keep it erect. The instability is reduced by increasing the number of longitudinal compartments, and by pumping out pairs of compartments in succession, so as to have as small a free water surface in compartments as possible. The pontoon should have

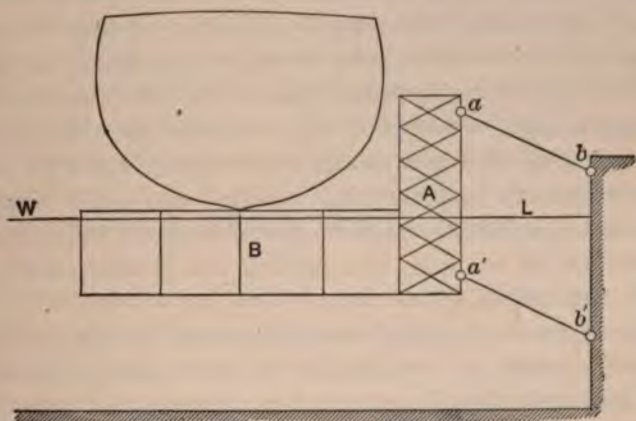


FIG. 109.

adequate transverse framing to keep it in form, but there is no advantage in making such transverse compartments water-tight.

The dock will be most unstable when the platform loaded with a large and deep ship is just below the surface of the water. The negative metacentric height may be calculated for the pontoon and the ship upon it (making allowance for the mobility of water in partially filled compartments) when in the position of greatest instability. The upsetting moment for a small inclination can be calculated by aid of this negative metacentric height, to serve as a basis for determining the forces acting on the linkage $aba'b'$ and consequently the requisite dimensions of the members of that linkage. The angle to which the pontoon may incline will depend on the rigidity of the frame A, the linkage $aba'b'$, and the pontoon itself, as well as on the looseness of the joints of the linkage. The angle chosen for calculation should be in excess of the probable angle due to lack of rigidity and looseness of the joints. Allowance

for inclination of the bar of the linkage must be made when making calculations for their size and strength.

In the Hoogla, near Calcutta, there is a hydraulic lifting dock which was designed to meet the difficulty of getting a proper foundation in the shifting sands. Two rows of columns are embedded in the sands with a sufficient distance between them to receive a large ship. Each column has inside it a long hydraulic ram lifting upward. Between the two rows of columns is a platform to receive the ship, with double links at each column suspending the platform from the heads of the hydraulic rams. The platform can be lowered deep enough to receive a ship, and after the ship is in place and properly blocked and shored the hydraulic rams are pumped up simultaneously and the ship is lifted above the water-level. Such a dock can have the platform plated in, thus forming a pontoon that may help lift the ship; or a pontoon can be interposed between the ship and the platform.

If the centre of gravity of the platform and the ship carried by it is below the heads of the rams, the whole structure of the platform and its loads will be stable, provided that the rams are properly supported and guided.

Floating Docks are used in places where it is difficult to get a proper foundation for a dry dock or a lifting dock; or sometimes

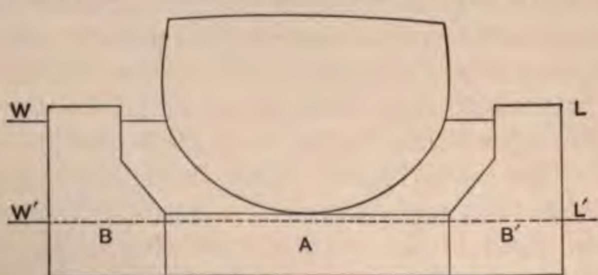


FIG. 110.

a floating dock is chosen because it is cheaper to build. Such a dock of the usual form consists of a shallow pontoon like A, Fig. 110, on which the ship may be lifted out of the water for painting or repairs, and two side pontoons BB' to give stability when the pontoon A is submerged deep enough to receive the ship and also during

the process of lifting the ship. The dock is lowered by admitting water to proper compartments, and is raised with the ship in place by pumping out the water. The amount of water to be pumped out may be only a little more than the displacement of the ship lifted, and this is much less than the water that must be pumped from a dry dock; it may be noted that a smaller ship in a dry dock requires a greater amount of water to be pumped out. It is claimed that a floating dock can therefore work faster than a dry dock, but that must of course depend on the pumps furnished each, and there is no reason except expense why the pumps for a dry dock may not be as large as desired. In either case the pumps are centrifugal pumps driven by directly connected engines, as the head to be overcome is not large and the use of the pumps is intermittent. The pumps and engines for a floating dock are placed in one of the side pontoons.

Some arrangement is always provided for painting a floating dock, more especially as it remains in harbor, where marine growths form rapidly. Some small docks have been so designed that they can be careened for that purpose; but this would require great depth of water for a large dock and might give rise to excessive stresses. A common way is to make the dock in three or more longitudinal pieces, and also to make the middle pontoon separable from the side pontoons; any part of the dock may then be separated and may be docked on the remainder of the dock. Another advantage comes from this arrangement in that the dock may be broken up into two or three small docks that can handle small ships.

A large dock recently built for the U. S. Navy has the following arrangement for painting: In addition to the arrangement for bolting the middle pontoon to the side pontoons in the normal position as indicated in Fig. 110, there are two others, one for bolting the middle pontoon 9 feet higher and another for bolting it 6 feet lower. The middle pontoon is, as usual, in several sections, and any one of these may be cut loose when the dock is lowered; when the depth is sufficient this section is bolted in the higher position and then the dock is pumped out and the section mentioned is lifted entirely out of the water and may be scraped and painted. The side

pontoons are also in sections, any one of which may be cast loose when the dock is lowered, and then the middle pontoon is bolted in the lower position, after which that section of the side pontoon can be lifted and painted.

A floating dock must be considerably wider than the largest ship to be lifted to give space on the deck of the middle pontoon for working, and also to give light and air. It may be somewhat shorter than the longest ship, since the support for the ship need not be carried to the extreme ends. It must have enough buoyancy to lift the heaviest ship and its own weight and give a freeboard of two or three feet to the deck of the middle pontoon. It must have sufficient stability to insure safety in all positions when lowered, when lifting a ship, and when raised with a ship in place. The most unfavorable conditions will occur when the dock is lifting a large ship with a high centre of gravity, and the most critical period is likely to be found just before the deck of the middle pontoon comes to the surface of the water. In that condition the deck with the ship in place should have a metacentric height of two feet, allowing for the mobility of water in any partially filled compartments. It is evident that the separation of the side pontoons from the middle pontoon and the division of the latter into sections may restrict the water that can reduce the stability by its mobility; if necessary, other water-tight compartments should be made by introducing longitudinal partitions in the middle pontoon. There should also be a regular routine for pumping out the various compartments to avoid instability and to keep the dock level. Some floating docks have the side pontoons widened at the bottom, as indicated by Fig. 110, to reduce the danger of instability at the most critical period before the deck of the middle pontoon emerges; but such a construction restricts the area of the deck and interferes with work.

A complete investigation of the statical stability of the dock when lifting a large and deep ship should be made by the usual methods, including a curve of metacentric heights and curves of statical stability, both for the position of least stability and for the normal position for working with the platform above the water-level. Cal-

culations should also be made to insure the ability to lift parts of the dock safely when it is necessary to clean and paint them.

In discussing the strength of a dock, it must be considered that the side pontoons or compartments are so deep and stiff that they will supply all the longitudinal strength of the dock. The platforms must, consequently, consist of a series of transverse beams or girders that transmit to the side compartments the loads due to the weight of the ship and the buoyancy of the water displaced by the dock. The longitudinal joints between the platform and the side pontoons must be substantially made and properly cared for. The transverse members of the platform should be continuous, and the longitudinal members should be sufficient to provide for water-tightness and to give general structural strength for the dock as a whole.

In calculating the strength of a dock, its greatest load may be assumed to be concentrated on a portion of its length near the middle; half or two-thirds of the length may be chosen for this purpose. The weight of the ship may be assumed to be uniformly distributed over this part of the dock, and the ends beyond the load may be assumed to be without load save from their own weight; at first sight it may appear as though we should draw a curve of weights for the ship much as is done in calculating the strength of the ship itself, but the extreme ends of the ship are not likely to be supported, and the ship will be strong enough and stiff enough to distribute its load nearly uniformly over its supported length. The dock will be assumed to be free from water-ballast, as will be the case when the ship is lifted. A floor-girder under the ship will be affected by three systems of loading: (1) a concentrated load equal to its portion of the weight of the ship; (2) a uniformly distributed load equal to its own weight; and (3) an upward buoyancy, which is also uniformly distributed.

The following discussion of the loads and bending moments acting on a floating dock is offered as a suggestion for a general method which will need modifications and additions in the applications to any special case. Let the displacement of the ship chosen for the computation be D tons, and let a floor-girder carry $\frac{1}{n}$ part

of the ship; then the concentrated load at the middle of the girder is

$$\frac{1}{n} D \text{ tons.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Assume 35 cu. ft. of sea-water to weigh a ton; let the length of a floor-girder be l inches, let the space between girders be s inches, and let the draught of the dock carrying the ship be h inches. Then the buoyancy acting on a girder will be

$$\frac{2240}{35 \times 12^3} lhs \text{ pounds.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the weight of a girder and of the longitudinal members, plating, etc., between two girders be W tons; then the distributed load on a floor-girder will be

$$\frac{2240}{l}\left(W - \frac{1}{35 \times 12^3} lhs\right) = w \text{ pounds per foot of length.} \quad . \quad (3)$$

This quantity will always be negative since the buoyancy exceeds the weight of the girder and attached parts, which will be indicated by the negative sign which the numerical result will always have. The supporting force at the end of the girder will be

$$F_0 = \frac{2240}{2}\left(\frac{1}{n} D + lw\right) \text{pounds.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This supporting force for a girder under the ship will be an upward force, since the weight of the ship and the portion of the dock under it will always exceed the buoyancy of that part of the dock; consequently the numerical value of F_0 will always have a negative sign.

Two cases may now arise depending on whether or not the floor-girders can be treated as fixed at the ends. In order that they may be treated as fixed, the middle pontoon must be efficiently bolted to the side pontoons and the latter must be maintained erect by some device like flying bridges from side to side over the top of the ship's hull. Perhaps it will be best to neglect the fixing in all cases in practice, but for illustration of methods both cases will be considered.

If the floor-girders are treated like a beam supported at the ends, the greatest bending moment will be found at the middle, and will be equal to

$$M = \tfrac{1}{2}lF_0 - \tfrac{1}{8}lw. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If, on the other hand, the floor-girders are treated as fixed at the ends, we may proceed as follows: The shearing force at the distance x from an end of a girder will be

$$F = F_0 + \frac{x}{l}w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The bending moment at the end may be represented by M_0, and the bending moment at the distance x from the end will be

$$M = M_0 + \int \left(F_0 + \frac{x}{l}w\right)dx.$$

$$\therefore\ M = M_0 + F_0x + \frac{x^2}{2l}w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The slope will be

$$\alpha = \frac{dv}{dx} = \frac{1}{EI}\int \left(M_0 + F_0x + \frac{x^2}{2l}w\right)dx. \quad . \quad . \quad . \quad . \quad (8)$$

$$\alpha = \frac{1}{EI}\left(M_0x + F_0\frac{x^2}{2} + \frac{x^3}{6l}w\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where v represents the deflection at any distance x from the end of the girder, and E and I are, respectively, the modulus of elasticity and the moment of inertia of the section of the girder. But the slope is zero at an end of the girder; consequently, making $x=l$, we have

$$0 = M_0l + \frac{F_0l^2}{2} + \frac{l^3}{6l}w.$$

$$\therefore\ M_0 = -\frac{F_0l}{2} - \frac{l}{6}w, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which determines the value of M_0; after which equation (7) may be used for determining the bending moments.

The bending moment at the middle, where $x = \tfrac{1}{2}l$, will be

$$M_m = M_0 + \tfrac{1}{2}F_0l + \tfrac{1}{8}lw. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The greatest bending moment, whether at the end or the middle, must be used for calculating the stress by the usual formula,

$$f=\frac{M_y}{I}. \quad \cdots \cdots \cdots \quad (12)$$

The deflection may now be obtained by integrating the equation (9), giving

$$v=\frac{1}{EI}\left(M_0\frac{x^2}{2}+F_0\frac{x^3}{6}+\frac{x^4}{24l}w\right), \quad \cdots \cdots \quad (13)$$

which will give for the middle, where $x=\frac{1}{2}l$,

$$v=\frac{1}{EI}\left(M_0\frac{l^2}{8}+F_0\frac{l^3}{48}+\frac{l^3}{24\times16}w\right). \quad \cdots \cdots \quad (14)$$

The several equations just deduced may be applied to floor-girders beyond the ship and which, consequently, have no concentrated load at the middle, provided that the supporting force is made equal to

$$F_0'=\frac{2240}{2}W. \quad \cdots \cdots \cdots \quad (15)$$

Coming now to a side compartment, it appears that it is affected by two uniformly distributed loads. The first is the weight of the ship, which is distributed over half or two-thirds of the length of the dock, and the second is the excess of the buoyancy over the weight of the dock, which is distributed over the entire length of the dock. Special weights, such as the pumping machinery and cranes or derricks, if there are any, will not be considered now; if such weights are relatively large, they may call for special consideration. It will be convenient to treat half this side compartment or side structure as a cantilever, as represented by Fig. 111. Let the downward load per inch of length where applied be w_1 pounds, while the upward load or buoyancy will be taken to be w_2 pounds per inch of length. The shearing forces at any section at a distance x from the end will be

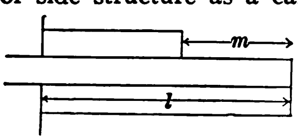


FIG. 111.

$$F=-w_2x, \quad \cdots \cdots \cdots \quad (16)$$

so long as x is less than m, the distance from the end of the dock to the load due to the ship. Near the middle of the dock

$$F = -w_2x + w_1(x-m). \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The corresponding bending moments are

$$M = -\tfrac{1}{2}w_2x^2,$$
$$M = \tfrac{1}{2}(w_1-w_2)x^2 - w_1mx + \tfrac{1}{2}w_1m^2. \quad . \quad . \quad . \quad . \quad (18)$$

At the middle of the dock the bending moment will be

$$M_0 = \tfrac{1}{8}(w_1-w_2)L^2 - \tfrac{1}{2}w_1mL. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It is, however, necessary to determine the location of the maximum bending moment, which may not be at the middle, by equating to zero the differential of the bending moment; that is, by making $F=0$ in equation (17). This gives

$$O = -w_2x + w_1(x-m).$$
$$\therefore \; x_m = \frac{w_1m}{w_1-w_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The value of x_m thus determined will give the maximum bending moment when substituted in equation (18). The stress is then to be calculated by means of the usual equation,

$$f = \frac{My}{I}.$$

Lifting and Depositing Dock.—A peculiar form of floating dock designed by Mr. Latimer Clark* is represented by Fig. 112. It consists of a deep girder A to which are secured a number of transverse pontoons B; stability, when the dock is immersed, is provided by the auxiliary pontoon C which is joined to the girder A by parallel rods

* Trans. Inst. Nav. Arch.

ab and *cd*. The girder *A* is a box girder as designed by Mr. Clark, but, as there is free communication from the inside of this girder to the sea, it might properly be made with an open framework. This girder carries the pumping machinery in closed compartments, which are large enough to float it. There appears to be no reason why the girder *A* should not be closed and made to contribute to the stability of the dock while the platform *B* is immersed; it could be opened to the sea when *B* is above the water, if desired. The pontoon is closed and is ballasted with cement; it does not aid in lifting the dock, but serves only to provide stability when the platform is immersed, and to give additional stability when the platform is above the water-level. The pontoon *C* is divided at convenient intervals to give free play to the parallel rods *ab* and *cd*, but the several parts are rigidly framed together.

The pontoons at *B* are long and narrow; one end of each is rigidly fastened to the girder *A*, and the other is free. There is considerable space between successive pontoons which in a manner resemble the fingers of the hand. This feature is provided so that the dock carrying a vessel may be drawn sideways between transverse rows of piling, properly capped with the timber, and then sunk so as to

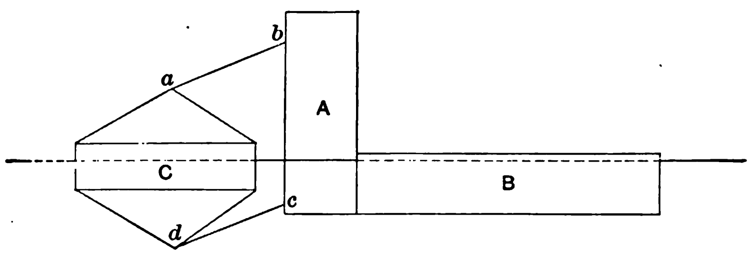


FIG. 112.

deposit the ship on the piling for cleaning and repairing. After the ship is cleaned and repaired it may be returned to the water by the dock. The designer claims that ships may be advantageously built on piling and lifted off when complete by such a dock without the danger and expense of launching. He also claims that warships can be laid up on piling when in reserve, instead of in a wet

basin. A dock of this design was built for the Russian government, which had a total length of 280 feet, but it was in three sections, two 100 feet long and one 80 feet long, so that it could be made of convenient length or the sections could be used separately. The girder, A, was 44 feet 6 inches high and 12 feet wide. The pontoons, B, were each 15 feet wide, 18 feet deep, and 72 feet long; the last dimension gives the width of the platform of the dock. The space between the pontoons B was five feet. It can raise a vessel with a displacement of 4000 tons in one hour. This dock was specially designed to handle certain circular iron-clad floating batteries which were from 100 to 120 feet in diameter. For this purpose the two longer sections were used separately, one being brought under each side of the vessel.

Careening.—Before dry docks and floating docks were common, the wooden vessels then in use, even if of large size, were careened or hove down to repair or clean the bottom. Even at the present time it may be convenient to heave down a small vessel at some remote place where there is no other way of making repairs. At any rate, it will be interesting to consider briefly the methods and conditions of careening vessels.

In preparing to heave down a ship it was customary to remove the ballast and all heavy weights, such as guns and stores. All weights not removed were made fast. Topmasts and yards were sent down, and all openings at which water could enter, such as ports, hatches, etc., were closed and made water-tight. Pumps were arranged so that they might draw from the lowest part of the hold in any position of the ship. Temporary platforms, from which the pumps could be worked, were arranged so that they could be kept horizontal.

The two principal masts (foremast and mainmast) were reinforced on one side by struts running from the upper part of the mast to the main deck. The topmasts were sometimes used for this purpose. The top ends of the struts were lashed to the mast, and the lower ends were supported on timbers which distributed the force over the deck, which in turn was supported from below by temporary pillars, running from deck to deck, under the main deck, and carried down to the ceiling of the hold. The shrouds on the sides opposite

the struts were reinforced by lines running from the mastheads to the end of spars thrust through convenient ports. The spars were made fast inboard and had their ends tied down to eye-bolts in the side of the ship.

Two barges were commonly used for heaving down a large ship, one for each principal mast (foremast and mainmast of a three-masted ship). A cable was passed round each barge amidships and formed into a loop, on the side nearest the ship, to give a point of attachment for the heaving-down tackle from the masthead of the ship. A line was sometimes made fast inboard, carried out of a port on the side away from the barge, passed under the keel, and connected through a tackle to the head of a mast on the barge, to serve as a preventer during the process of heaving down.

The process of heaving down may be divided into two periods; in the first the barges are drawn up against the side of the ship by the transverse component of the pull on the heaving-down tackle. Suitable fenders should be placed between the ship and barges to guard the side of the ship from injury during this process. The second period begins when the heaving-down line becomes vertical and the barge swings clear from the side of the ship. The stresses on the tackle, masts, struts and shrouds, and the stability of the ship during heaving down, can be investigated by assuming that a weight W_1, equal to the pull on the tackle, is applied at the masthead of the ship (Fig. 113); this, of course, applies only after the second period begins.

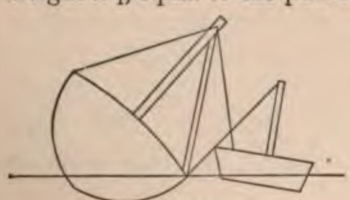


FIG. 113.

The centre of gravity of the ship when stripped and reinforced for heaving down may be determined by an inclining experiment. Let G be the centre of gravity, and let D be the displacement of the ship in this condition. If a weight W is added at the head of the mast, the displacement will become $D+W$, and the centre of gravity will rise to G_1, Fig. 114, determined by the equation

$$GG_1 = \frac{HG \times W}{D+W}.$$

If G_1 is above the metacentre corresponding to the water-line W_1L_1, after the weight is added, the ship will take an inclination that can be determined as follows:

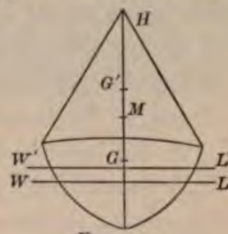


FIG. 114.

Let $aa'a''$ (Fig. 115) be the curve of statical stability for the displacement $D+W$ and with the centre of gravity at G. The effect of raising the weight W from G to H, that is, of raising the centre of gravity from G to G_1, will be to decrease the righting moment by the amount (see page 88)

$$GG_1 \sin \theta.$$

This will be represented by the curve of sines $bb'b''$. The intersections of the curves $aa'a''$ and $bb'b''$ at e and e' will show the positions of equilibrium under the given conditions. The first position of equilibrium will be stable and the second will be unstable. The part of any ordinate, as $a'b'$, intercepted between the two curves will show the righting moment for the given angle of inclination with the load W at the head of the mast. These intercepts laid off as ordinates will give the curve of stability with the weight W applied. A repetition of this process for a sufficient number of assumed values for W will enable us to draw a curve $cc'c''$, Fig. 115, showing the moments required to give any inclinations. Such a curve should logically begin at the inclination where the barge swings clear of the ship, but it is convenient to draw the full curve. The ship will become unstable where the curve of righting moments crosses the axis of abscissæ, and beyond that point the ship will require to be supported by the line passing from the barge under the keel of the ship (see Fig. 113). This line should, therefore, be kept taut during the process of heaving down, but with little stress on it till the ship shows signs of becoming unstable, after which the heaving-down tackle will naturally become slack. The weight W in the preceding discussion is really the sum of the pulls on the two masts; the pull of the tackle attached

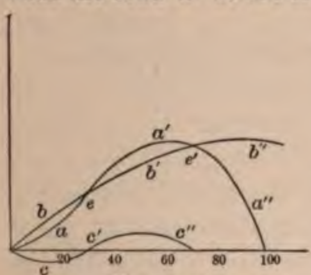


FIG. 115.

to the mainmast is, of course, the larger, and it will be well to make this tackle strong enough to take the entire pull with a fair margin of safety, since the division of the forces between the two tackles cannot readily be determined. The tackle for the foremast may, of course, be somewhat lighter.

A ship may be hove down beside a pier, leading the tackle from fixed points on the pier, provided the level of the water in which the ship floats remains unchanged. The work may be carried on in water affected by the tide, and advantage can be taken of a rising tide to aid in the work, but the process then becomes more delicate and will require much care.

CHAPTER VII.

LAUNCHING.

THE successful launching of a ship depends so much on the slope and stability of the launching-ways that the discussion of launching may well begin with a description of the construction of building-slips, that is, of the foundation on which the ship is built and on which the launching ways are laid. When it is considered that the displacement of a ship when carrying a full cargo may be more than 30,000 tons, and that a ship when ready for launching may weigh from one-third to half as much, the importance of a sufficient foundation for building and launching is evident. A slight yielding of the ways during launching may make the ship stop and cause much trouble and expense, and any considerable deflection may cause disaster.

Building-slips.—Shipyards are sometimes located on ground that is so firm that the building-blocks may be laid directly on the ground, or on transverse timbers that rest on the ground; small vessels are often built on such temporary blocking. It is to be borne in mind that blocking laid on the ground is liable to be disturbed by frost unless the soil is porous and well drained; this may be of no importance for a vessel small enough to be completed in the summer months.

Some kind of foundation is always laid for slips on which large vessels are built. The ground conveniently located for building ships is likely to be loose or wet, and the foundation can, in such case, be most conveniently made by piling. The piles are driven in transverse rows close together under the keel and more widely located at the sides, and may be capped by cross-balks, which are preferably of hard wood so that they shall not be indented by the tops of the piles. These cross-balks are commonly framed on to the tops of

the piles which, of course, reduces the bearing surface; if they are of soft wood like fir, it appears better to saw the piles square across and to secure the cross-balks by iron fastenings. The individual cross-balks are laid horizontally to receive the building-blocks; the assembly of them which forms the surface of the building-slip is given a slope up from the water, which may be the same as the slope of the blocks and the keel, except that near the bow of a long ship it is found more convenient to stop the piling near the surface of the ground and get the required height by extra blocking. The ground between the piling and nearly up to the level of the cross-balks may be filled with gravel or other convenient material.

The construction thus briefly described is that most commonly found in private yards, but more substantial work is sometimes put in. For example, the New York Shipbuilding Company has put in a very substantial piling uniformly over the entire front where ships are built, and can consequently lay down a ship wherever it may be convenient on that front. The shore end of this building front is carried on masonry arches beneath which are passageways for transporting materials under the bows of the ships building. The slope given to this building front is equal to that to be given to the launching-ways, that is, the slope is considerably more than that given to the building-blocks, and consequently the stems of ships building are high above the surface of the building front.

Navy-yard slips are sometimes built of masonry in the most substantial manner. Piles are driven, if necessary, and are bound together by transverse and longitudinal timbers, all of which are kept well below the finished surface of the slip. The piles and timbering are now filled in and covered with beton, on the surface of which is laid a pavement of large stones. Cross-pieces of oak are laid in the pavement at intervals of 6 or 7 feet and project above its surface, to receive the building-blocks.

The building front of the Fore River Ship and Engine Company is a bed of compact gravel which gives a good natural foundation. Their building-slips have a foundation of large blocks of granite capped with concrete, which is laid in wide horizontal steps on which the blocking can be conveniently laid.

Blocking for small ships may be of soft wood, but hard wood

should be used for large ships. The blocking should give at least three feet below the top of the keel for fastening and calking wooden ships, and at least four feet for riveting steel ships.

Slope.—The surface of the blocking on which the keel is laid has a slope of $\frac{1}{12}$ in certain French navy yards, and a slope of $\frac{1}{14}$ to $\frac{1}{16}$ in English navy yards. Private yards sometimes have a slope of $\frac{1}{32}$. The slope of the launching-ways is usually greater than the slope of the blocking, especially when the building-slip has a moderate slope. Care must be taken that the bow of the ship shall not touch the ground during launching. A large slope of the launching-ways gives greater certainty that a ship will start and not stop during launching, and in some other ways reduces the dangers of launching, but it gives a high velocity to the ship as it enters the water which may be difficult to control, and requires a greater depth near the water's edge.

As the building of a ship progresses the weight of the ship is borne mainly by the blocking under the keel, but also to some extent by shores, and sometimes by buttresses built of blocking under the bilges.

Ship-houses.—Building-slips in some yards are housed over. This practice probably arose when wooden ships were allowed to season in frame. They have the advantage that there is less interruption of work from the weather, especially by snow and ice. They have the disadvantage of excluding light and air to a considerable extent.

Methods of Launching.—Ships are usually built end on to the water front, and are launched by the stern; occasionally a ship has been launched by the bow. This method gives the best advantage of the water front, which is frequently restricted. In shipyards on the Great Lakes which have a water front on a narrow river ships are commonly built broadside to the water front and are launched sidewise. The launching-ways are then given a large inclination.

In England and America end launching is always carried out with two launching-ways placed under the bilges. The width between the ways from centre to centre may be one-fourth to one-third of the beam of the ship. In France a variety of methods is used. Sometimes there are two ways, as in the English practice.

Sometimes three ways are used, one under the keel and one under each bilge. Sometimes the ship is launched on the keel; in such case there are two side ways, but they are merely to steady the ship. The pressure on them is light, or there may be no pressure if the ship is well balanced.

Launching-ways are made of thick, hard-wood planking, which is drawn into place under that longitudinal member of the hull which can best resist the concentration of loading during launching. Small ships which have no bilge keelsons or other convenient longitudinal members are seldom subjected to severe strains in launching.

The ways are laid on the cross-balks which cap the piling, or are blocked up from them or from the surface of the slip. They are bolted to the cross-balks and are kept to the proper gauge by the distance-pieces and tie-rods, and are shored up at the sides. A ribbon of hard wood fastened to the ways gives a side bearing for the bilge-logs and insures that the ship shall not leave the ways. The slope of the launching-ways varies from $\frac{1}{12}$ to $\frac{1}{24}$, and is usually a little greater than the slope of the building-blocks. The slope sometimes increases toward the water, and in that case the ways are arcs of large circles.

The Bilge-logs are made of hard wood, usually oak, in several lengths, which are tied together by chains. The width of the bilge-logs should be such that the pressure shall be between two and three tons per square foot. If the pressure is more than three tons, the lubricant is liable to be squeezed out, and if it is less than two tons, the ship may not start when released.

On top of the bilge-logs packing is filled in up to the skin of the ship amidships, and as far forward and aft as the sections are full. When the transverse sections become fine forward and aft, the filling may be made up of timbers on end called poppets. They have their feet stepped on the bilge-log, and the heads are caught under the edges of strakes of plating, or are secured under iron fittings temporarily fastened to the skin of the ship. The heads of the poppets may be faced with iron when necessary.

The forward poppets are bound together at the head by an iron rod, and chains are passed under the forefoot of the ship to give a bearing when the ship lifts at the stern and pivots at the bow.

The bilge-logs are tied together with a chain at the fore end to prevent spreading when the ship pivots; this chain is made so that it can be disengaged by men on the deck of the ship after the ship is afloat, and the bilge-logs may be towed ashore separately.

Shoring up Inside.—If there is any possibility that the structure of the ship will be strained during the launching by the concentration of the weight on the ways, the weight may be distributed by shoring up inside the ship. This may be done by fitting wooden pieces between the inner and outer skins, when there is a double bottom, and by placing temporary wooden pillars between the bottom and the lower deck, and over these other pillars between decks. Or the stresses may be distributed by inclined struts forming, with the framing of the ship, a series of trusses.

There is, however, a difference of opinion as to whether the internal shoring is necessary or beneficial. Small vessels will not require shoring. There appears to be more necessity for shoring at the forefoot over the chains on which the ship pivots.

After the launch the ship should be inspected and bulkheads should be tested with water-pressure when possible, to see if any members have been bent or twisted, and if any rivets have been started.

Lubrication of Ways.—In England the packing is fitted in place two or three weeks before the launching, and is then removed, the bilge-logs are turned out of the ways, and the ways are coated with good tallow and soap. The bilge-logs are then put in place and the packing is definitely placed. On the morning of the launching the ways are oiled through holes made for that purpose. For the launch of a ship of 7400 tons displacement there was used in one case 4 tons 6 cwt. of tallow, 5 cwt. of soft soap, and 55 gallons of oil; the last being applied just before launching. The U. S. S. *Oregon* at launching weighed about 4000 tons, and there were used 930 pounds of stearine and 1090 pounds of soft soap; the U. S. S. *Olympia* weighed 2400 tons, and there were used 490 pounds of tallow and 585 pounds of soft soap; both were launched at San Francisco in cool weather. The amount and quality of grease used depend largely on climatic conditions and vary with the location of the yard and the season of the year.

Launching.—Shortly before the ship is to be launched the weight of the vessel is taken up on the bilge-logs by driving wedges under the packing, and the shoring and blocking are removed progressively, beginning aft; but some blocks, three to twenty, are left at the bow to hold the ship till the releasing device is ready to act. Cleats are also used to fasten the after-ends of the bilge-logs to the ways till the time for releasing the ship.

The launching should proceed promptly after the ship is wedged up, as the lubricant is liable to be squeezed out and the ship may not start when released.

Releasing Devices.—Various devices are employed for letting the ship go at the right time. They should hold the ship firmly and certainly till the proper time and then shou d let go easily and quickly. A favorite releasing device is in the orm of two dog-shores which have their heads toward the bow caught under a piece fastened to the bilge-log. Their feet are planted against projections from the ways. The heads and the bearings for them may be covered with iron to prevent indentation which would make the release uncertain. The two shore-dogs may be knocked down by dropping weights on them at the proper instant. The weights are suspended over the shore-dogs, and to make sure that they are released simultaneously they may be hung at the ends of a rope which passes over pulleys to a convenient location where the rope can be severed at the middle.

The releasing device for launching the *Oceanic* was a pivoted trigger which had its tail held up by a hydraulic ram. When the pressure on the ram was released, the trigger dropped and released the ship.

A primitive but reliable method of releasing the ship is to saw through the sliding-ways, which are extended and fastened securely at the shore ends. The planks in which the ways terminate should be of straight-grained oak which is not likely to split or splinter when the ship starts, and the men who saw them must keep clear so as not to be hurt. The two planks must be cut at the same rate so that they may part at the same time and let the ship start squarely. This method is better adapted to starting a ship which is launched on one way under the keel, and which has consequently only one plank to sever.

Starting Devices.—In case the ship should not start when released she should be started by some proper device. It will be convenient to have a hydraulic ram or jack at the end of each bilge-log, and one or more bearing against the stem. Sometimes levers are rigged near the middle of the ship; the short arm of each is arranged to bear on a projection on the bilge-log, and the arm has a tackle by which a large force may be applied by a gang of men.

Check-rope.—A hawser may be fixed to the stern of the ship and attached to a chain cable, and an anchor carried some distance up-stream, and arranged to cant the stern up-stream when the ship is clear of the ground, and so prevent danger that the ship may run ashore on the other side of the stream. Tugs are in readiness to take the ship when launched and tow it to the berth.

Checking the Velocity.—When large ships are launched in narrow waters, especially if the ways have a large inclination, there is more or less difficulty in controlling the ship if the full velocity due to launching is developed. The velocity may be checked in various ways. A wooden shield may be fixed at the stern which will offer a large resistance to the motion of the ship through the water; since it begins to act as soon as the ship reaches the water, it may act too soon and endanger the success of the launching. Two hawsers may be carried from the bows of the ship and laid so that they may drag along the ground till the ship is free, and then check her. To mitigate the shock of snubbing the vessel, the hawsers are carried aft beyond the bitts and lashed at intervals to another cable stretched taut on the deck; these lashings are torn away as the ship is checked, and bring her to rest quietly.

A device that has found favor in some of the Scotch yards is the converse of this. Hawsers are made fast on the ship and carried over the bows and fastened to heavy chains that are laid in piles at intervals. The weight of the chains and the friction of dragging them over the ground effectually check the ship; the piles of chain can be arranged to act when required.

A very complete system was provided for checking the velocity of the U. S. S. *Chattanooga*,* which was launched in the Kill von

* M. S. Chace, Trans. Nav. Archs. and Marine Engs., Vol. 11.

Kull where the effective width was only once and three-quarters the length of the ship. Two 15-inch manila cables were carried from the bows and secured at intervals by 7-inch manila stops to chain cables fixed at the sides of the launching slip. As an additional means of checking the ship, a shield with a surface of 93 square feet was fixed at the stern. It was estimated that the ship at launching might have a maximum weight of 1700 tons, and that the velocity might be 10 feet per second, so that the kinetic energy to be absorbed might be 5900000 foot-pounds. It was found by tests in the laboratories of the Massachusetts Institute of Technology that the stops, which were 20 feet long, each absorbed 3000 foot-pounds when broken. To give a good margin, 50 stops on each side were provided. The stops began to act when the ship left the ways, and 29 on the starboard and 24 on the port side were broken, bringing the ship to rest in 146 feet after the first stop broke.

Launching on the Keel.—When a launching-way is placed under the keel, both the way and the shoe on the keel must be made in short pieces, because the blocking under the keel cannot be removed from any considerable length at one time. The way must be fastened to the cross-balks, and the pieces must be well fastened together; for this purpose iron plates at the edge of the pieces may be used, or continuous plates of iron may be placed at each edge and bolted through and through. Lauching on the keel has the advantage that there is less liability of trouble if the way should yield. If there are two ways, and one yields a little, the ship is thrown out of the vertical and the whole launching-cradle is racked and may jam. On the other hand the weight of the ship is concentrated on one way, when the ship is launched on the keel, which will give excessive loads unless the foundation is very secure. Launching on the keel and on bilgeways, that is, on three ways, distributes the weight more than either of the other two methods, but it is difficult to obtain an even distribution.

Launching Sideways.—Some of the shipyards on the Great Lakes have for their water front a narrow river which will not allow of launching by the stern, not even if a diagonal direction across the river should be chosen. It has become the custom to launch ships sideways in these yards. This method has also been adopted in

isolated instances on our seaboard and in England. The notable failure in the first attempt to launch the *Great Eastern* sideways has given this method of launching a bad repute which it does not deserve, since in that case the failure was probably due more to lack of adequate preparation than to the method used. Launching by the stern will, however, be preferred when there is sufficient water, as it gives a better use of the water front, as has already been suggested.

The preparation of the ground for building ships that are to be launched sideways is much the same as usual, except that the ship is on an even keel when building, and must be set high enough to give a good slope for launching. A large number of ways is used, and as the weight is well distributed, soft wood like pine or fir may be used. The launching-ways are laid between the building-blocks, and a cradle is fitted for each way. The ways are lubricated and the ship is wedged up as usual.

The releasing device consists of two large wooden levers, bearing on fixed fulcrums. The short ends catch one of the cradles near the bow and another near the stern. The long ends of the fulcrums are held by ropes that are cut when the ship is released. The blocking is removed before the launch, and all obstructions are removed from the path of the ship. In launching, the bow is released just before the stern.

The launching-ways are sometimes extended a little below the surface of the water, but not infrequently they stop short of the water. Formerly some builders made the outer ends of the ways so that they would pivot as the ship neared the water, and give a sharper incline at that time; this has since been abandoned.

The ships launched in this way have usually little rise of floor, and are partly water-borne before the keel reaches the water. They heel over to a considerable angle, and there is a large wave thrown up with a good deal of splashing, but there does not appear to be much risk of injury.

Outer End of Slip.—The permanent ways for launching by the stern are frequently too short for safe launching, and in such case a temporary prolongation must be made. Very often this temporary construction may be made in the form of a timber cribwork laid

on the bottom. As the ship is partially water-borne when it gets on to the prolongation of the ways, the pressure on them is reduced, and they need not be made as stable as the permanent ways. Some yielding, if it does not give the ship an inclination, is of little importance.

Operation of Launching. — The launching may be divided into three distinct periods.

(1) The ship slides down the ways with accelerated velocity toward the water. This period is short and calls for little comment. The water at high tide, when the launch is made, very often comes up to the stern-post, and then this period does not exist.

(2) The ship is partly water-borne, but rests on the ways the entire length of the bilge-logs. The centre of the reaction of the ways on the bilge-logs approaches the bow as the stern is more and more immersed. If the ways are too short, the ship may tip over the ends of the ways. Should this occur while the ship is moving slowly, there will be a concentration of pressure against the bottom of the ship which will tend to buckle the plates and distort the framing. If there is any current, as from the flow of the tide, past the end of the slip, the ship may be slewed round and serious damage may occur. If the ship has a large velocity, contact between the ship and the ways may be lost momentarily and the ship may pound on the ways; if this action is severe, it may injure the structure of the ship.

(3) When the centre of reaction of the ways on the bilge-logs reaches the fore-poppet, the stern rises and the ship pivots about the fore-poppet; from this instant the ship touches the ways only at that place. If the ways should prove too short during this period, the bow will slip off and the ship will dip. If the bow does not strike the ends of the ways and if there is sufficient depth of water at the end of the ways so that the bow will not touch bottom, no inconvenience will come from this action.

Statics of Launching.—It is convenient to treat the problem of launching as if the ship moved down the ways very slowly, that is, to make it a problem of statics; afterwards the effect of the velocity may be considered. The velocity of the ship on the ways depends on so many circumstances which can only be partially controlled that the dynamics of launching are uncertain. Fortunately the danger liable to arise if the ways are short is mitigated to some

extent by the velocity of the ship down the ways; consequently the length of the ways may be determined by statical methods only.

Statics of the First Period.—During the first period, before the ship reaches the water, the centre of the reaction of the ways may be assumed to be under the centre of gravity of the ship; it would, of course, be exactly there if the ways were quite true, and if the wedging up of the packing on the bilge-logs were uniform. The weight of the ship, applied at the centre of the reaction of the ways, may be resolved into two components, one at right angles to the ways, producing pressure on the ways, and the other parallel to the ways, causing the ship to slide. If the slope of the ways is $\frac{1}{m}$, then the similarity of the triangles in Fig. 116 gives the following proportion between the weight of the ship W, the reaction of the ways R, and the force F urging the ship down the ways:

$$W:R:F::\sqrt{1+\frac{1}{m^2}}:1:\frac{1}{m}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But m is never less than 12, and is usually greater, consequently W is never more than $\frac{3}{10}$ per cent greater than R; and we may always neglect the slope of the ways in discussing moments and reactions during the launching of a ship.

In order that the ship may start when released, the ratio of the force F to the pressure R should be greater than the coefficient of friction of the ways lubricated with tallow. This coefficient is somewhat greater than the coefficient of friction after the ship has started, which coefficient will be considered under the dynamics of launching. It is asserted that ships have started with a slope of $\frac{1}{32}$ for the launching-ways, which corresponds with a coefficient of friction of 0.03; so small a coefficient for starting cannot always be depended upon.

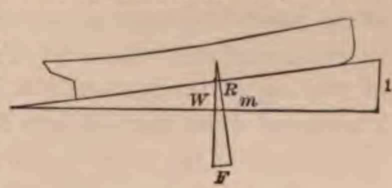


Fig. 116.

Statics of the Second Period.—The second period of launching begins when the ship enters the water; usually the heel of the sternpost enters the water first.

At any instant there are three forces acting on the ship: (1) its weight W (Fig. 117) acting downward through its centre of gravity G; (2) the reaction of the ways R acting at right angles with the ways; and (3) the buoyancy of the immersed part of the ship acting upwards at the centre of figure B of that portion. There are also the several forces which tend to accelerate or retard the ship, but consideration of such forces will be reserved for the dynamical discussion of launching.

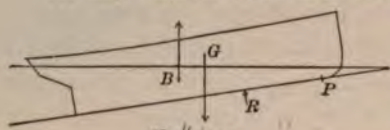


FIG. 117.

The weight of the ship W at the time of launching, and the centre of gravity, are determined from the amount of material worked into the ship and its location. When possible the material should be weighed under the direction of some competent person; it is well to continue this throughout the building of the ship.

The buoyancy of the immersed portion of the ship can be computed from the lines of the ship; most conveniently by aid of Bonjean's curves.

If the slope of the ways is neglected, the weight, buoyancy, and reaction form a system of three parallel forces, of which the first two are known, and the third may be determined by taking moments about a convenient axis.

Curves of Displacements and Reactions.—In the investigation of the several important events of the second period it is convenient to refer everything to the edge of the water, that is, to the intersection of the top of the launching-ways by the water-level at the height anticipated when the ship is to be launched; the effect of variations of the height of the water and of other conditions will be investigated later. The important events of this period are tipping, pivoting, and leaving the ways. It is customary to determine the points on the ways at which these events take place or are liable to take place by aid of a diagram.

In Fig. 118 the origin of coordinates O is taken at the water's edge, and abscissæ are distances measured down the ways from the water's edge, not allowing for the slope of the ways. On this diagram three curves are located: (1) a curve of displacements, (2) a curve of moments, and (3) a curve locating the reaction.

The curve of displacements gives the displacement of the immersed part of the ship for each position of the stern-post; thus the diagram which was drawn for the launching of the U. S. S.

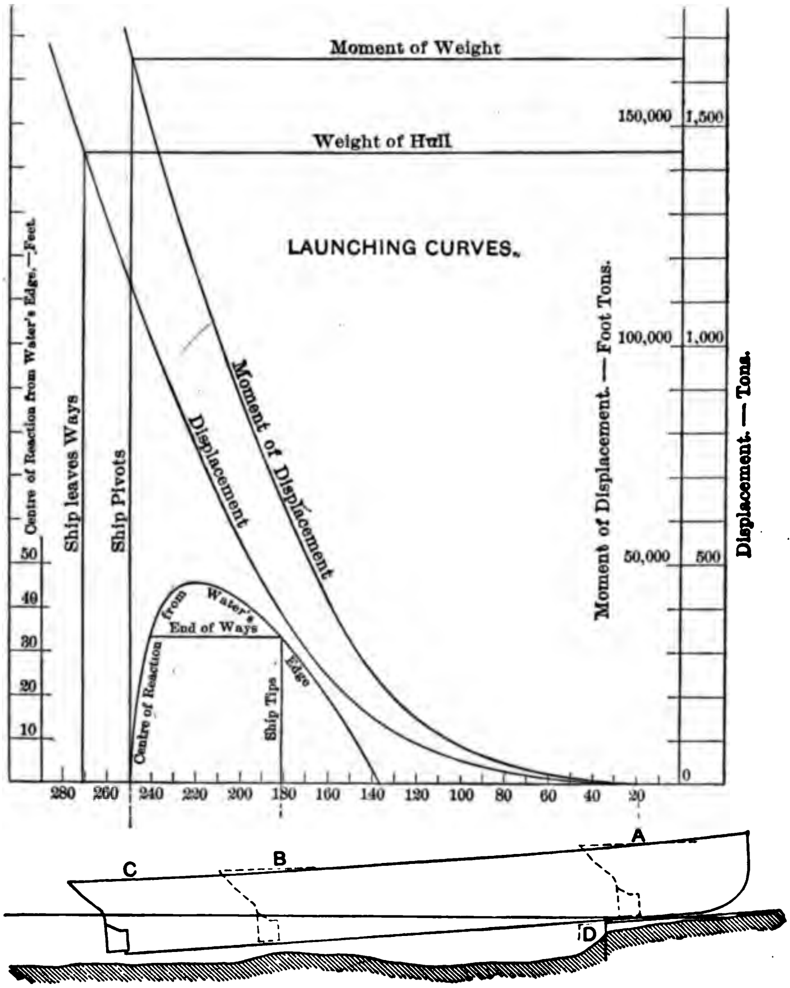


FIG. 118.

Chattanooga shows that when the stern-post was 160 feet from the water's edge the displacement was 245 tons. Now the launching weight of this ship was 1435 tons, which weight is represented by a horizontal line on the diagram. The difference between the

launching weight and the buoyancy is the reaction, so that this latter force may be measured directly on the diagram; if desired, a curve of reactions can be drawn on the diagram; it is omitted in the figure to avoid complication. The diagram shows that when the stern-post was 160 feet below the water's edge the reaction of the ways was 1190 tons.

The curve of moments gives the moments of the displacement about an axis at the fore-poppets; its scale is ten times that for the displacement, as indicated at the right of the diagram. On the diagram there is drawn a line to represent the constant moment of the weight of the hull about the poppets.

The third curve shows the distance that the reaction of the ways is *below* the edge of the water. To get the ordinate of this curve, take the abscissa which represents the distance the stern-post is below the edge of the water, and on the ordinate at that abscissa measure the reaction from the curve of buoyancy to the line which represents the launching weight; also measure the resultant moment about the poppets from the curve of moments to the line which represents the constant moment of the hull; the resultant moment divided by the reaction gives the distance of the reaction from the forward poppets; knowing the distance of the stern-post from the edge of the water, and the distance of the forward poppets from the stern-post, we readily find the distance of the reaction from the edge of the water and lay it off as the ordinate. Thus the resultant moment is 121,000 foot-tons, and the reaction is 1190 tons when the stern-post is 160 feet below the edge of the water; consequently the reaction is then 101 feet below the forward poppet. But the forward poppet is 244 feet forward of the stern-post, and consequently the reaction of the ways is

$$160-(244-101)=17 \text{ feet}$$

below the edge of the water.

Tipping.—When the ship is at the beginning of the second period with the stern-post at the edge of the water, the reaction of the ways is assumed to be directly under the centre of gravity of the hull. As the ship enters the water the point of application travels down the ways to a certain maximum distance *below* the edge of the

water, and then begins to return up the ways. In Fig. 118 the curve of distance of the reaction from the water's edge begins when the reaction is at the edge, and is continued to the position at which the ship begins to pivot on the poppets; the interesting part of the curve is at and near its maximum, and in practice only that part of the curve will be drawn. The maximum distance on Fig. 118 is 46 feet, and the stern-post is then 220 feet from the edge of the water. The point *D* shows the length which the ways should have had as determined by this method. But the actual length of the ways was only 33 feet below the edge of the water, and consequently for a very slow launch the ship might be expected to tip over the end of the ways at the position indicated by dotted lines at *B*. While the ship is sliding the distance of 50 feet from 180 to 220 feet below the edge of the water, the pressure on the ways is concentrated near the end of the ways, tending to indent the bottom of the ship and also to stop the ship. If the effect is sufficient to check the ship, and if at the same time a strong current is running past the stern, the ship may swing round and be seriously damaged. Most commonly the ship slides down rapidly, as happened for the launching in question, and the danger from tipping is thereby reduced. Experienced naval architects know from experience what allowance can be safely made for velocity and what length must be given to the ways. Sometimes the ship loses contact momentarily with the ways and then strikes against them. A repetition of such an action may make the ship pound on the ways just before it pivots on the poppet, and if the pounding is severe, damage may be done to the ship. Very frequently the ways are deliberately made shorter than would be required to avoid tipping, if, in the opinion of the naval architect, the ship is sure to have a good velocity of launching, and if he thinks this will reduce the danger of tipping.

Naval architects in England and America very commonly investigate the danger of tipping by taking moments of the weight of the ship and of the buoyancy of the immersed carene about an axis at the end of the ways. A diagram is thus drawn with distance of the centre of gravity down the ways for abscissæ and with moments for ordinates, and two curves are drawn, as in Fig. 119, to represent the moments of buoyancy and the moments of the weight of the ship; the

latter is a straight line as shown. If the curve of moments of buoyancy is always the higher, then the ship will not be in danger of tipping. In fact the excess of the moment of buoyancy over the moment of weight for any position of the centre of gravity of the ship gives a sort of margin of safety. Should the line which represents the moment of the weight of the ship cut the curve of moments of buoyancy, then the ship will begin to tip over the end of the ways when its centre of gravity reaches the distance from the end of the ways, shown by t, Fig. 119. From there on the ship will drag over the end of the ways, being inclined more and more as the tipping moment increases, and then righting as the moment decreases, until at t', where the curve of buoyancy is cut the second time, the ship will again lie with the cradle in contact with the ways.

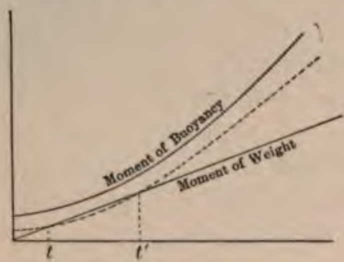


FIG. 119.

Pivoting.—So long as the moment of the weight of the ship about the poppet is greater than the moment of the buoyancy about the same point, the cradle will remain in contact along the ways; when these moments become equal the reaction of the ways will be directly under the poppet and the ship will begin to pivot on the chains at the poppets. To find the point at which this will occur, note on Fig. 118 the intersection of the curve of the moments of displacements with the line of the moments of the weight. The corresponding abscissæ (250 feet on the figure) is the location of stern-post when the ship pivots: the location of the ship when pivoting is shown by the full-line elevation at C. The location of the poppet at that time is determined by measuring from the location of the stern-post. Thus the poppets are 244 feet from the stern-post for the ship under discussion, and consequently pivoting occurs when the poppets are 6 feet below the edge of the water.

Very commonly the two positions of the ship, one for the maximum distance of the reaction below the edge of the water, and the other at which pivoting begins, are not far apart. If the position for pivoting should be reached first, there will, of course, be no danger of tipping.

Pressure on the Poppets.—It is very important to know the pressure on the poppets at pivoting, as that is the greatest concentrated force acting on the ship during a normal launch. This pressure is the reaction at pivoting and can be measured directly on Fig. 118, and for the ship under discussion is 300 tons.

Statics of the Third Period.—During the third period the ship is in contact with the ways under the fore-poppets only; the reaction under the fore-poppets and the angle of equilibrium for any given position of the ship on the ways may be determined by a method similar to that given on page 184 for grounding on the keel. There does not seem to be any sufficient reason for going through this work, as the reaction diminishes gradually from the instant the ship begins to pivot till it leaves the ways. It is, however, important to determine the point of contact of the forefoot on the ways at the instant when the ship leaves the ways. This may be done by finding the point on the ways where the depth of water is equal to the draught of the ship

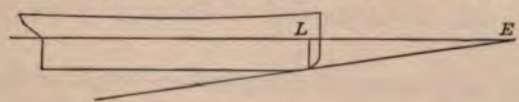


FIG. 120.

measured at the fore-poppets to the bottom of the bilge-log, as in Fig. 120. If the draught is d feet, and if the inclination of the ways is $\frac{1}{m}$, then the distance of the point in question from the edge of the water is md. Even though the ways should be somewhat shorter than thus indicated for the purpose of giving support till the ship leaves the ways, there is little chance of trouble provided that there is depth of water enough to prevent striking when the ship dips after it leaves the end of the ways. As in the discussion of tipping, it may be said that the dynamic actions due to velocity of the ship down the ways mitigate the ill effects due to short ways.

Effect of Variations.—In the determination of the several important points of launching a ship it has been assumed that the launching weight, the location of the centre of gravity, and the level of the water are all known. There is always some uncertainty about all three, and especially the last, which on a seaboard depends on the height of the tide and on the direction and force of the wind.

It is, therefore, important to determine beforehand the effect of any probable variation in each of the several conditions.

In the first or direct investigation of tipping, and the investigations of the points of pivoting and of leaving the ways, all are measured from the actual edge of the water. Whether the water be high or low, if the height of the tide varies from that assumed, all three points move up or down the ways with the edge of the water.

If the English method of investigating tipping is used, then the form of the curve of moments of buoyancy, Fig. 119, changes with the tide and a new curve must be drawn. The line representing the moment of weight is unchanged.

If the weight of the ship at launching changes, or if the location of the centre of gravity varies from the point assigned to it, or if both of these changes occur, then the curve of reaction from water's edge (Fig. 118) must be redrawn. But if the English method is used, it will be sufficient to draw a new line for the moment of the weight of the ship about the end of the ways, at the proper angle on Fig. 119.

The effect on pivoting of changing the weight of the ship or of shifting its centre of gravity may be determined by drawing a new horizontal line on Fig. 118 to represent the constant moment of the weight about the fore-poppet.

To find the point at which the ship leaves the ways, if the weight or the location of the centre of gravity changes, the method shown by Fig. 120 may be repeated, using the proper draught.

Dynamics of the First Period.—During this period, before the ship enters the water it slides down the ways under the influence of the component F of the weight (Fig. 121), which is parallel to the ways. If ϕ is the coefficient of friction, then the resistance of friction is

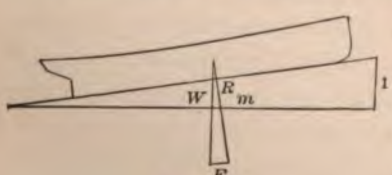


FIG. 121.

$$\phi R = \phi \frac{W}{\sqrt{1 + \frac{1}{m^2}}} = \phi W \text{ approximately,} \quad . \quad . \quad . \quad (2)$$

because $\frac{1}{m^2}$ is very small.

If a is the acceleration of the ship, the force that is required to impart this acceleration is

$$a\frac{W}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the force parallel to the ways is

$$F = \phi W + a\frac{W}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But from the proportion (1), page 217,

$$F = W\frac{1}{m\sqrt{1+\frac{1}{m^2}}} = \frac{W}{m} \text{ approximately;} \quad . \quad . \quad . \quad . \quad (5)$$

and if this value is inserted in equation (4) and the resulting equation is solved for ϕ,

$$\phi = \frac{1}{m} - \frac{a}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If x is the distance that the ship has slid down the ways in the time t, then

$$a = \frac{d^2x}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The methods of measuring the times and distances during a launching will be considered later.

The coefficient of friction on the ways depends on the kind and quality of lubricant, on the temperature, and on the pressure per square foot of the ways. Only pure materials of the best quality should be used; poor or improper material is likely to lead to trouble and expense.

Hauser* gives the following particulars of the friction on the ways. The ships appear to have been launched on their keels, and the lubricant is known by the generic name *suif* (tallow).

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* Construction Navale.

COEFFICIENT OF LAUNCHING FRICTION.

Name of Ship.	Weight at Launching, Tons.	Length of Sliding-way, Feet.	Breadth of Sliding-way, Feet.	Load, Tons per sq. ft.	Coefficient of Friction.
Admiral Baudin........	3684	311.3	2.62	4.50	0.0217
Furieux...............	1668	237.8	1.97	3.56	0.0257
Pomvoyeur.............	490	181.7	1.31	2.05	0.0324
Flamme................	412	164.0	1.54	1.62	0.0370
Etoile................	254	150.9	1.38	1.29	0.0407
Pluvier...............	281	167.3	1.54	1.03	0.0400
Alcyon................	165	150.1	1.38	.79	0.0487
Sentinelle............	33	65.6	.99	.50	0.0532

Hauser says further that the coefficient of friction of the *Admiral Baudin* at starting (including the effect of four blocks left under the forward end of the keel) was 0.04055, which is nearly double that given in the table, as deduced from the velocity of the ship.

Inspection of the preceding table shows that the coefficient of friction for launching decreases as the pressure increases, as is always the case for lubricated friction. Small vessels, which are likely to give a light pressure per square foot of launching-ways, must have a large inclination of the ways to make them start freely.

The following particulars of the launch of a tug are interesting, as they are very near the limit on account of the small weight per square foot of the ways:

TUG FEARLESS.

Date of launch......................Nov. 7, 1891
Weight of vessel........................200 tons
Area of sliding-ways...............194 square feet
Weight per square foot......................1 ton
Angle of launching-ways..................1 in 15

The vessel started slowly, and, though it did not stop, it was difficult at times to detect motion. The launch was assisted by men hauling on ropes.

A test by Messrs. Eastwood and Patch* on the steam-yacht *Pantooset*, which had a designed displacement of 680 tons and which weighed 303.6 tons when launched, gave a coefficient of friction of .038; the slope of the ways was $\frac{7}{8}$ inch in 1 foot, and the pressure per square foot was 1.09 tons.

* Thesis, Mass. Inst. Tech., 1902.

Dynamics of the Second and Third Periods.—In the discussion of launching it has been assumed that the ship moves so slowly down the ways that the dynamic effects due to velocity can be neglected; this sometimes happens even when every precaution is taken, but usually the ship enters the water with considerable velocity and throws up a wave. The reaction of the water under the stern acts lik eadded buoyancy and hastens the pivoting on the fore-poppets, and there is also less danger of tipping over the ends of the ways. Thus it appears that the dynamic effects tend toward safety and that it is possible to launch on shorter ways than are called for by statical requirements.

Velocity of Launching.—It is important to determine the velocity of the ship down the ways to see if the inclination of the ways is proper, and to determine the coefficient of friction. The coefficient of friction should properly be determined from the velocity during the first period before the ship touches the water, but as the ship usually has only a short distance to travel in this period, and frequently is launched when the tide is high enough to reach beyond the stern-post, the calculation for coefficient of friction is made from the velocities at points after the ship has entered the water a little distance. The coefficients thus obtained gradually diminish, but the correct values can be inferred from a curve with distances for abscissæ, and the corresponding values of the coefficient for ordinates.

A variety of methods with varying degrees of refinement have been used for measuring the velocities of ships down the launching-ways. A simple way is to have a series of observers with stop-watches at stations down the ways, who note the transit of some easily recognized mark on the hull. Mr. Simonds * has devised a modification of this method by which the observations can be taken by two persons. In preparation a number of marks at intervals are painted on the hull, and a stop-watch (or any watch with a long second hand) has a narrow ring of paper pasted on its face. The two observers take their stand near the edge of the water, and as the marks on the hull pass the station one observer gives a signal and the other observer marks the position of the second hand with a pencil; if the launching takes more than one minute, the second round

* Proc. Inst. Nav. Arch.

of the hand is marked on the other edge of the paper. This method is much improved if the record is made on a chronograph on which seconds are marked by a clock or chronometer, and the transits of marks on the hull are made by closing an electric circuit. To determine the velocity of the *Pantooset* a number of large spikes were driven into the sliding-ways which closed an electric circuit for a simple chronograph; the times were indicated by a chronometer which closed the same circuit every half-second. There is an objection to driving spikes into the sliding-ways, as they may catch on some accidental obstruction. To obviate this difficulty a cord or wire may be attached to the bow of the ship and dragged down by it during the launch. A heavy pendulum beating seconds (or any known interval of time) may carry a brush loaded with paint and mark the times on the cord, which is afterwards to be stretched in the same way and measured. The launching velocities of the U. S. S. *Texas* and *Raleigh* were determined by a chrono-

LAUNCHING DATA, LEVIN SHIPYARD.

	Bancoora.	Diana.	Kerbela.	Quetta.	Clyde.	India.	Antonio Lopez.	Makina-pua.	Bucarest.	Goorkha.
Date, 1880 to 1882.	N'v. 19	Sept. 3	Oct. 6	Mch. 1	Jun. 15	Aug. 27	Nov. 8	D'c. 31	D'c. 20	Mch. 7
Length, feet......	323	160	285	380	390	390	370	160	260	390
Beam, "	40	26	35	40	42	42	42	$26\frac{5}{8}$	37	42
Depth, "	$28\frac{1}{2}$	9	$26\frac{1}{2}$	$20\frac{1}{2}$	34	31	30	$13\frac{1}{16}$	24	31
Draught at launching.	$7\frac{3}{8}$	$2\frac{3}{4}$	$8\frac{1}{4}$	$9\frac{3}{4}$	$10\frac{1}{2}$	9	$9\frac{1}{2}$	$4\frac{1}{8}$	$7\frac{1}{2}$	$9\frac{1}{4}$
Displacement tons	1560	162	1245	2075	2585	2125	1944	266	1108	2187
Length standing-ways..........	343	......	366	409	414	433	392	251	268	406
L'gth sliding-ways	246	114	209	288	300	298	268	112	186	300
Breadth ways, ins..	23	15	23	23	23	23	23	15	16	23
Area, square feet..	943	285	801	1104	1150	1142	1027	280	496	1150
Tons pe sq. foot .	1.65	0.56	1.55	1.87	2.25	1.86	1.89	0.95	2.20	1.90
Inclination, 16ths of an inch per foot	8 to 12	8 to 20	7 to 15	8 to 16	7 to 13	7 to 12	9 to 11	8 to 18	6 to 12	8 to 12
Temperature, Fah..	......	......	......	......	60	56	52	37.5	40	56
Maximum velocity, feet per second..	15.3	13.7	17.7	16.5	16.4	14.7	16.6	13.7	14.4	12
Distance run at maximum veloc.	183	170	247	235	280	260	210	170	140	220
Unresisted velocity at this point....	23.7	26.4	28.4	29.1	28.2	26.4	25.3	23.2	19.7	24.3
Time to maximum velocity, seconds	28	27	33	30	34	35	30	36	38	67
Time of leaving ways..........	40	30	40	43	43	49	44	43	48	84

graph on which distances were indicated by a wire which rotated a drum and closed an electric circuit for each revolution; the times were indicated by a chronometer; and events like starting, pivoting,

and leaving the ways were recorded by an observer who could close the same circuit.

Having the times and distances observed during the launching of a ship, a diagram like Fig. 122 may be plotted; the times from

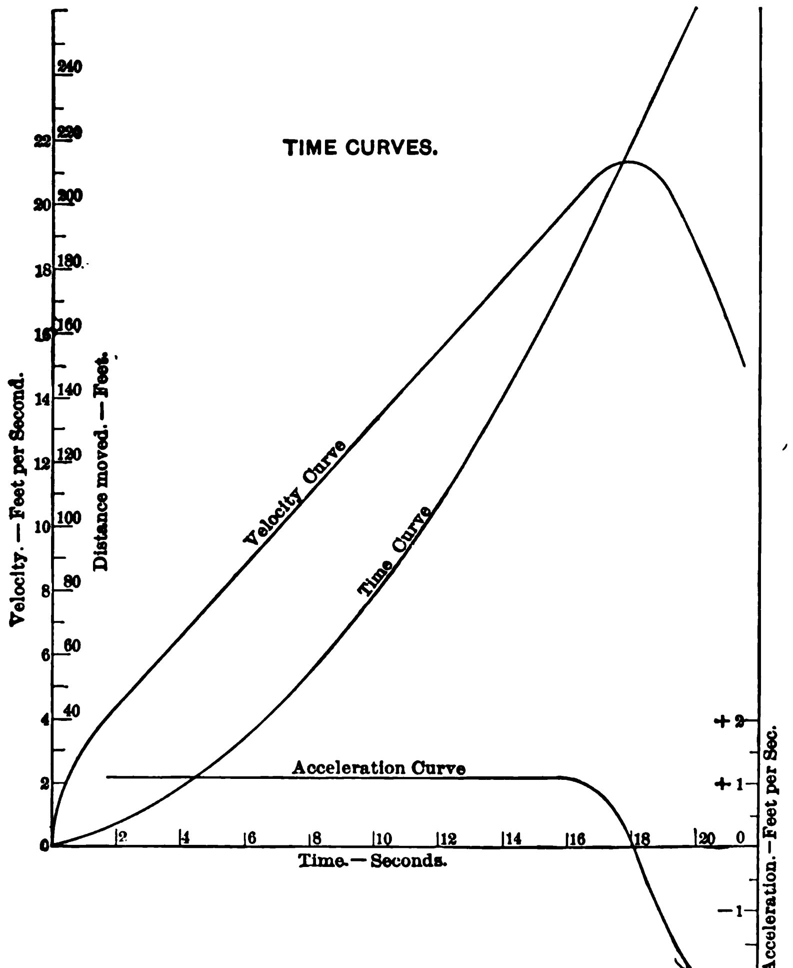


FIG. 122.

the instant of starting are laid off as abscissæ, and distances down the ways (neglecting the slope) are erected as ordinates and a smooth curve is drawn; the diagram gives the results of observa-

tions by Messrs. Cross and Hilkin* during the launching of the U. S. S. *Chattanooga.* The ordinates of a differential curve constructed by the method explained on page 25 give the velocities down the ways, and the ordinates of a second differential curve give the accelerations. The curve of accelerations is liable to be irregular and unreliable on account of the unsatisfactory method of graphical differentiation. After the start the acceleration of the ship is likely to be uniform, or nearly so; consequently we may fair the curve of accelerations, and, starting from the faired curve, may integrate twice by aid of the integraph or the planimeter. The integral curve thus obtained should represent the observations of times and distances down the ways; if it is not unsatisfactory, the curve of accelerations may be modified and the work repeated until a satisfactory result is obtained.

Launching Data.—Mr. William Denny † gives in the table on page 228 the data for the launching of a number of ships at the Levin Shipyard, Dumbarton. The table below gives data for the

LAUNCHING DATA, U. S. NAVAL VESSELS.

	Texas.	Raleigh.	Olympia.	Oregon.
Date	28 June, '92	31 Mch., '92	5 Nov., '92	24 Oct., '93
Length, feet	$301\frac{1}{3}$	300	344	348
Beam, feet	$64\frac{1}{16}$	42	52	$69\frac{1}{4}$
Depth, feet	$39\frac{3}{4}$	$33\frac{7}{16}$		
Draught forward, feet	9	$6\frac{1}{16}$	$6\frac{1}{2}$	$11\frac{1}{4}$
Draught aft, feet	12	$12\frac{1}{3}$	$15\frac{5}{16}$	$12\frac{1}{2}$
Metacentric height, feet	10	$3\frac{3}{4}$		
Launching weight with cradle, tons	2449	1322	2434	4162
Length of ground-ways, feet	$381\frac{3}{4}$	$336\frac{1}{2}$	396	383
Length of sliding-ways, feet	215	$205\frac{1}{4}$	274	276
Width of ways (effective), inches	26	13	$17\frac{3}{8}$	$35\frac{1}{8}$
Area of ways, square feet	932.7	447.7	$793\frac{1}{2}$	$1621\frac{1}{3}$
Pressure on ways, tons per square foot	2.62	2.97	3.06	2.57
Inclination of ways, 16ths inch per foot	12.7	12	10.6	8.9
Camber of ground ways, inches	6	4	0	12
Depth at end of ways, feet	$12\frac{3}{4}$	$8\frac{1}{8}$	$7\frac{1}{2}$	$8\frac{1}{3}$
Distance stern-post to water, feet	10	$\frac{1}{2}$	1	0
Time of launching, seconds	49	58		
Time to maximum velocity, seconds	16	23		
Time to pivoting, seconds	23.2	23.7		
Maximum velocity, feet per second	18.6	18.8		
Distance to maximum velocity, feet	154	182		
Distance to pivoting, feet	300	253		
Coefficient of friction initial	0.026	0.051		
" " " mean	0.02	0.048		

* Thesis, Mass. Inst. Tech., 1903.

† Trans. Inst. Naval Arch., Vol. XXIII.

launching of several U. S. naval vessels.* In connection with the latter it is interesting to note the time occupied by the operation of launching as given in the following table for the *Olympia* and the *Oregon:*

TIME REQUIRED FOR LAUNCHING, MINUTES.

	Olympia.	Oregon.
Ramming up	31	33
Interval	14	20
Removing shores	5	17
Interval	13	14
Removing part of keel-blocks	16	33
Interval	25	3
Removing keel-blocks and letting go	25	35
Total working time	77	118
Total time of launching	129	155

Stability during Launching. — During the first and second periods, while the ship is supported by the ways, its stability is insured provided the ways do not yield.

After a ship which is launched on the keel begins to pivot, it is supported in part by the reaction at the forefoot and in part by the buoyancy of the water on the immersed portion. The investi-

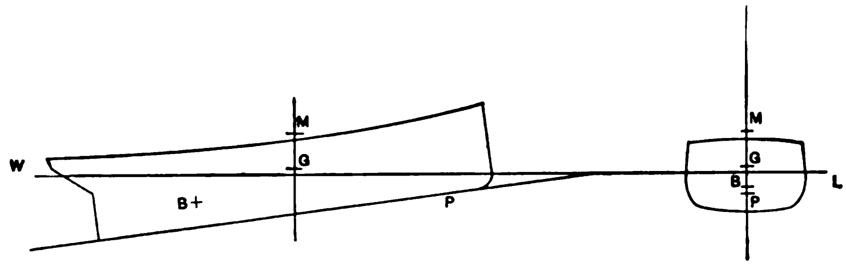


FIG. 123.

gation of the stability may be made as follows: In Fig. 123 let the ship be supposed to be at the position where pivoting begins. Let G be the centre of gravity of the hull where the weight W of the hull is applied; let B be the centre of buoyancy of the immersed portion, and let P represent the forefoot. Draw the transverse section of the ship through G, as shown in the figure, and project on to this

* Proc. Soc. Nav. Archs. and Marine Engs., Vols. 2 and 8.

section the points G, B, and P. Let WL be the corresponding water-line. If the ship takes a transverse inclination, it will be about a horizontal axis through P, since we may assume that the forefoot cannot leave the ways.

The effort of the buoyancy will be applied at the metacentre M, on the vertical BM through B, and at a height above B.

$$r_p = \frac{i_p}{V_p},$$

in which i_p is the transverse moment of inertia of the water-line WL, and V_p is the displacement of the immersed portion.

For a slight inclination θ we shall have for the righting moment about the axis through P

$$D_p\, MP\theta - WGP\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which D_p is the displacement of the immersed part of the ship. Dividing by W and θ,

$$MP \cdot \frac{D_p}{W} - GP,$$

a factor which may be considered to correspond to the metacentric height of a ship which is afloat.

A ship which is launched on two ways will receive stability from the cradle after pivoting, and if it has a fair metacentric height when afloat is in little danger of tipping over sidewise.

Stability after Launching.—The stability of the ship after launching should be determined either by the metacentric method or by drawing curves of stability, as may appear to be advisable. A failure to attend to this matter may cause serious trouble, or may lead to disaster.

The displacement of the ship when launched may be estimated from the plans and working drawings, or may be determined more exactly from the record of materials worked into the ship. The position of the centre of gravity can be determined from the working drawings only, and is less likely to be satisfactory. The centre of gravity of merchant-ships is likely to be higher when launched than when finished and ready for sea. If the ship is full and has

a flat floor, the height of the metacentre above the centre of buoyancy after launching is likely to be large, as will be seen from the expression

$$r=\frac{i}{V},$$

in which i has a value approaching that for the ship at load draught, while V is small. Such a ship may have a large metacentric height at the launching displacement, equal to or greater than the metacentric height at the load draught. But in general the metacentric height at launching is likely to be the smaller, especially if the ship is fine and has considerable rise of floor.

If the metacentric height of a ship at launching is not enough to allay apprehension, the entire curve of stability should be drawn. A ship is so light when launched that it usually has a good freeboard and may show a fair curve of stability with a small metacentric height. Professor Biles * gives an instance of a ship which was launched with a metacentric height of only 0.66 of a foot, but there was a good freeboard and all weights were secured, so that the launch was successful though the ship rolled over to an inclination of twenty degrees. Normand expresses the opinion that the metacentric height should be at least two feet, and that ballast should be added if necessary to give such a height; but the addition of ballast will never be made to a ship when launched unless there is conclusive evidence of danger of instability.

Professor Biles gives as causes that may give a ship an inclination during launching: (1) waves thrown up by the ship in narrow and unsymmetrical waters; (2) wind-pressure on the sides of the vessel; (3) inequality of time in releasing the sliding-ways and cradles; (4) unequal resistance of devices for checking velocity.

Marine Railways.—Ships which have not a large displacement are commonly hauled up on marine railways for painting and repairing. The ways, or rails, of such a railway have commonly a greater slope than launching-ways; the cradle for receiving the ship is mounted on wheels and is drawn up and let down by a rope or chain over a windlass. The cradle is let down with keel-blocks

* Trans. Inst. Engrs. and Shipbuilders of Scotland, Vol. XXXII.

in place, and the ship is floated on at high tide; as the tide falls the ship takes the blocking much as it would in a dry dock except that there is an inclination given to it; after the ship is in place and shored up the cradle is hauled up. When the painting and repairs are finished the ship is lowered down on the flood-tide and is released as the tide rises. The ways must be long enough to support the cradle when in its lowest position, and consequently there is no danger of tipping over the ends of the ways, and in general the operation is more under control and therefore less risky than launching.

Some modern ship-railways are given a large inclination, so that they may lift large ships in a comparatively short space. The cradle has a level surface to receive the ship, and the whole operation is more like docking than hauling up.

CHAPTER VIII.

THEORY OF WAVES.

THE surface of the water which is subjected to the action of the wind is thrown into waves of varying form and size; if the wind has recently changed and a cross-sea is running, the water takes complicated and evanescent forms which defy even approximate analysis. Under the action of a long-continued and steady wind waves are more regular, and have nearly the same height and length. After the wind ceases to blow and secondary surface disturbances have disappeared the sea is for some time affected by a long smooth swell which has a comparatively simple form.

Ordinary observation shows that it is the undulation that moves along as crest follows crest, for floating bodies move up and down and back and forth, remaining in the same general location. There is usually a slow drift of such floating bodies in the direction in which the wind blows, and that drift persists after the wind falls. There is no completely satisfactory theory of waves, but certain equations can be produced which conform to the laws of hydrodynamics and which give results that are confirmed by such observations as have been made on the lengths and velocities of waves.

A general treatment of hydrodynamics is long and difficult, and calls for certain methods and functions that are not commonly used by writers on engineering subjects. Fortunately almost all the problems that are interesting to the naval architect can be treated with two dimensions, that is, they are problems in plane geometry calling for only ordinary methods of analysis. By thus simplifying the investigations from the beginning it is possible to give a satisfactory and comparatively brief treatment of the problems usually considered by naval architects; those who desire to have a more complete knowledge of the subject should read some good modern

treatise on hydrodynamics, after having familiarized themselves with the requisite methods and functions.

A fluid may be defined as an aggregation of molecules which yield to the slightest effort to separate them from each other if it be continued long enough. Fluids are either liquids or gases; the former, with which we shall now concern ourselves, are only slightly compressible, and we shall consider that they are entirely incompressible.

A perfect fluid is defined as one which is incapable of sustaining any tangential stress or action in the nature of a shear; it can be shown that the consequence of this property is that pressure at any point of a perfect liquid is equal in all directions whether it be at rest or in motion. All known fluids can offer some resistance to tangential stresses; this property is known as viscosity, and it gives rise to an action in the nature of friction. In the discussion of waves friction will be neglected; that is, we shall consider that we have to do with an incompressible frictionless liquid.

Pressure at a Point.—The pressure at a point of a liquid at rest is the same in all directions whether the liquid is viscous or not. This is a direct consequence of the condition that liquids yield to the slightest force if long continued, and especially to a continued shearing force. A way of proving the proposition is as follows: In Fig. 124 let *abcd* be a section (partially external) of a rectangular parallelopiped having the dimensions Δx, Δy, Δz. The figure cut off by the inclined section will have the same volume and weight as the original parallelopiped, and if the angle of inclination is θ, the area of the section will be $\Delta x \Delta y \div \cos\theta$. Assuming the pressure on the upper surface to be uniform and equal to p pounds per square foot, the total pressure on that surface will be

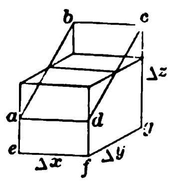


FIG. 124.

$$p\,\Delta x\,\Delta y \div \cos\theta,$$

and its component perpendicular to the base will be

$$p\,\Delta x\,\Delta y.$$

If the pressure on the lower surface is p' pounds per square foot, the total upward force will be

$$p'\,\Delta x\,\Delta y.$$

If the weight of the liquid per cubic foot is w, the weight of the figure *abcd* will be

$$w\,\Delta x\,\Delta y\,\Delta z,$$

which is to be added to the downward component of the pressure on the upper face to find the total downward force. For equilibrium we must have

$$p\,\Delta x\,\Delta y + w\,\Delta x\,\Delta y\,\Delta z = p'\,\Delta x\,\Delta y,$$
$$\therefore\ p + w\,\Delta z = p';$$

and if the dimensions of the figure approach zero, we shall have at the limit

$$p = p'.$$

The resultant forces under consideration are in a plane parallel to *dcgf*, from the symmetrical construction of the figure; but the pressure per square foot on the base and the area of the base are unchanged by changing the direction of its sides; consequently the section *abcd* may be made to take all angles and aspects, and thus the demonstration can be made general.

And further, while the only extraneous force considered is gravity, a little consideration will show that we will get the same conclusion for any vertical force. And again, by turning the figure on its side with *dcgf* horizontal, the proposition may be extended to include the action of horizontal extraneous forces.

If we consider that the figure in Fig. 124 is affected by a downward acceleration α, the force required to produce this acceleration will be

$$\alpha\frac{w}{g}\Delta x\,\Delta y\,\Delta z;$$

and as this force will be equal to the difference between the downward and upward forces acting on the liquid bounded by the figure *abcde*, we shall have

$$p\,\Delta x\,\Delta y + w\Delta x\,\Delta y\,\Delta z - p'\Delta x\,\Delta y = \alpha\frac{w}{g}\Delta x\,\Delta y\,\Delta z.$$
$$\therefore\ p + w\Delta z = p' + \alpha\frac{w}{g}\Delta z,$$

and at the limit $p = p'$.

Again, we may give the section *abcd* all inclinations and aspects by changing the angle θ and the direction of the base, and may

take account of the horizontal forces and accelerations by turning the figure onto a side with *dcgf* horizontal.

The pressure at a point can be equal in all directions for a liquid in motion only under the condition that it has no viscosity or friction. That condition will be assumed for all the problems which will be discussed in the present chapter.

Euler's and Lagrange's Method.—There are two methods of treatment of problems in hydrodynamics, both due to Euler, but one of them is commonly named for Lagrange, who gave much attention to its development.

In Euler's method, which is also called the flux method, attention is fixed upon a particular point in space (located by the rectangular coordinates x, y, and z), and the changes of pressure and velocity at that point are noted.

In Lagrange's method attention is given to a particular element of liquid (which may originally have the coordinates x_0, y_0, and z_0) and to its path, which is indicated by the coordinates x, y, and z. This latter method is, in general, difficult in application; it happens to be convenient for the discussion of the common or trochoidal theory of waves.

Equations of Equilibrium.—In dealing with the theory of waves it is convenient to consider the liquid as bounded by two vertical planes one foot apart, and extending in the direction of the motion of the waves. The only extraneous force is gravity, which acts vertically downward. Consider the condition of a horizontal parallelopiped, Fig. 125, having the dimensions $\varDelta x$, $\varDelta y$, and one foot, the last dimension being the distance between the bounding planes. Suppose that the pressure increases upward and toward the right. The increase of pressure per foot toward the right will be

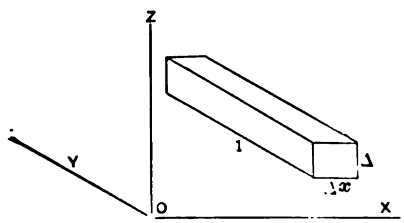


FIG. 125.

$$\frac{\partial p}{\partial x},$$

which is a partial differential coefficient, since the rate of increase may be variable. The increase of pressure in the distance Δx is

$$\frac{\partial p}{\partial x}\Delta x,$$

and the excess of pressure on the right face is

$$\frac{\partial p}{\partial x}\Delta x\,\Delta z.$$

This force will be equal to the mass of the parallelopiped multiplied by its horizontal acceleration. The mass is

$$\frac{w}{g}\Delta x\,\Delta z,$$

where w is the weight of a cubic foot and g is the acceleration due to gravity. The horizontal acceleration is

$$-\frac{d^2x}{dt^2},$$

which has a negative sign since the acceleration is toward the left. Consequently

$$\frac{\partial p}{\partial x}\Delta x\,\Delta z = -\frac{w}{g}\frac{d^2x}{dt^2}\Delta x\,\Delta z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The excess of pressure on the upper side of the parallelopiped over that on the lower side will be

$$\frac{\partial p}{\partial z}\Delta x\,\Delta z.$$

If the parallelopiped were affected by pressure only, it would have a vertical acceleration

$$-\frac{d^2z}{dt^2},$$

and, on the other hand, if it were affected by gravity only, it would have a downward acceleration, g, which has a negative sign since

distances are measured upward from the origin. The resultant vertical acceleration is, therefore,

$$-\left(g+\frac{d^2z}{dt^2}\right).$$

Equating the excess of pressure to the force required to produce the resultant acceleration,

$$\frac{\partial p}{\partial z}\Delta x\,\Delta z=-\frac{w}{g}\left(g+\frac{d^2z}{dt^2}\right)\Delta x\,\Delta z. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Simplifying equations (1) and (2),

$$\frac{\partial p}{\partial x}=-\frac{w}{g}\frac{d^2x}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\partial p}{\partial z}=-\frac{w}{g}\left(g+\frac{d^2z}{dt^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These are known as Euler's equations of equilibrium.

In Lagrange's method the coordinates x and z of a point at the time t are functions of the original variables x_0 and z_0. To get the proper equations of equilibrium for this method from equations (3) and (4), we make use of the equations

$$\frac{\partial p}{\partial x_0}=\frac{\partial p}{\partial x}\frac{\partial x}{\partial x_0}+\frac{\partial p}{\partial z}\frac{\partial z}{\partial x_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{\partial p}{\partial z_0}=\frac{\partial p}{\partial x}\frac{\partial x}{\partial z_0}+\frac{\partial p}{\partial z}\frac{\partial z}{\partial z_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

for changing the variables. To apply equation (5) we may multiply equation (3) by $\frac{\partial x}{\partial x_0}$, and equation (4) by $\frac{\partial z}{\partial x_0}$ and add; and in like manner we may multiply by $\frac{\partial x}{\partial z_0}$ and $\frac{\partial z}{\partial z_0}$ and add, to apply equation (6). This gives

$$\frac{\partial p}{\partial x_0}=-\frac{w}{g}\left\{\frac{d^2x}{dt^2}\frac{\partial x}{\partial x_0}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial x_0}+g\frac{\partial z}{\partial x_0}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

and

$$\frac{\partial p}{\partial z_0}=-\frac{w}{g}\left\{\frac{d^2x}{dt^2}\frac{\partial x}{\partial z_0}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial z_0}+g\frac{\partial z}{\partial z_0}\right\}, \quad . \quad . \quad . \quad . \quad (8)$$

which are known as Lagrange's equations of equilibrium.

Sometimes it is convenient to identify an element by some other means instead of using the original coordinates x_0 and y_0; for example, in the discussion of trochoidal waves an element may be identified by aid of the coordinates a and b of the centre about which the element revolves, as will appear when we come to that case. Evidently the method used for transforming equations (3) and (4) into (7) and (8) will give for any parameters a and b (by which an element can be identified)

$$\frac{\partial p}{\partial a}=-\frac{w}{g}\left\{\frac{d^2x}{dt^2}\frac{\partial x}{\partial a}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial a}+g\frac{\partial z}{\partial a}\right\}, \quad . \quad . \quad . \quad (9)$$

$$\frac{\partial p}{\partial b}=-\frac{w}{g}\left\{\frac{d^2x}{dt^2}\frac{\partial x}{\partial b}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial b}+g\frac{\partial z}{\partial b}\right\}. \quad . \quad . \quad . \quad (10)$$

Equations of Continuity. — Our conceptions of the nature liquids require that they shall be continuous without voids, and that the density shall be constant. If we concentrate our attention on a given space through which a liquid may be moving, then the weight of liquid in that space is constant. On the other hand, if we follow the motion of a certain mass of liquid, its volume must remain constant. These conceptions are sometimes used directly as here stated; commonly they are embodied in equations known as equations of continuity. Consider the space bounded by the parallelopiped in Fig. 126 terminated at the ends by planes one foot apart and having the dimensions Δx and Δz. The velocity of the water at the left-hand face is

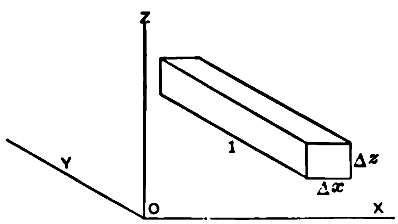


FIG. 126.

$$u=\frac{dx}{dt},$$

and in the time Δt the volume of fluid

$$\frac{dx}{dt}\Delta z\,\Delta t$$

will enter through that face. The increase in velocity per foot toward the right is

$$\frac{\partial}{\partial x}\frac{dx}{dt},$$

so that the velocity at the right-hand face is

$$\frac{dx}{dt}+\frac{\partial}{\partial x}\frac{dx}{dt}\Delta x.$$

In the time Δt a volume of liquid

$$\left(\frac{dx}{dt}+\frac{\partial}{\partial x}\frac{dx}{dt}\Delta x\right)\Delta z\,\Delta t$$

will flow out at the right-hand face. The resultant horizontal flow will be

$$\frac{\partial}{\partial x}\frac{dx}{dt}\Delta x\,\Delta z\,\Delta t.$$

In like manner the resultant vertical flow will be

$$\frac{\partial}{\partial z}\frac{dz}{dt}\Delta x\,\Delta z\,\Delta t.$$

Since the volume of the parallelopiped is unchanged and the density of the liquid is constant, the sum of the resultant flows must be zero, so that

$$\frac{\partial}{\partial x}\frac{dx}{dt}+\frac{\partial}{\partial z}\frac{dz}{dt}=0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This equation is frequently written

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Equation (11) or (12) is called Euler's equation of continuity.

Consider the condition of an elementary parallelopiped of liquid which had at the time t_0 and at the point (x_0, z_0) the transverse section $\Delta x_0 \Delta z_0$ and the length of one foot. At the time t this element will be at the point (x, z), where x and z are functions of x_0 and z_0. The rate of change of x with x_0 is $\frac{\partial x}{\partial x_0}$, so that the increment Δx_0 increases the abscissa by the amount $\frac{\partial x}{\partial x_0}\Delta x_0$. But the change in x_0 produces also a change in z_0 having the rate $\frac{\partial z}{\partial x_0}$, and the increment Δx_0 is accompanied by an increment $\frac{\partial z}{\partial x_0}\Delta x_0$. Consequently a motion of the point (x_0, z_0) to $(x_0+\Delta x_0, z_0)$ produce a motion of the point (x, z) to $\left(x+\frac{\partial x}{\partial x_0}\Delta x_0, z+\frac{\partial z}{\partial x_0}\Delta x_0\right)$. In Fig. 127 lay off $AF=\frac{\partial x}{\partial x_0}\Delta x_0$ and $FD=\frac{\partial z}{\partial x_0}\Delta x_0$; then AD is the resultant motion due to the increment Δx_0. In like manner an increment Δz_0 will produce the motions

$$AE=\frac{\partial z}{\partial z_0}\Delta z_0 \quad \text{and} \quad EB=\frac{\partial x}{\partial z_0}\Delta z_0.$$

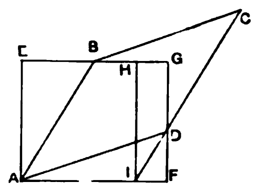


FIG. 127.

Complete the parallelogram $ABCD$ and the rectangle $AEGF$, and produce CD to I, and then draw HI parallel to GF. The area of the parallelogram $ABCD$ is equal to the area of the rectangle $AEHI$; consequently

$$\text{Area } ABCD=AF\,.\,AE-AE\,.\,FI.$$

But from similarity of triangles

$$AE:EB::DF:FI; \quad \therefore FI=EB\,.\,DF\div AE;$$

consequently

$$\text{area } ABCD=AF\,.\,AE-EB\,.\,DF. \quad . \quad . \quad . \quad . \quad (13)$$

But since liquid is incompressible, the volume of the original elementary parallelopiped must be equal to that of a parallelopiped having

the section $ABCD$ and the length of one foot. Replacing the quantities in equation (13) by their values already obtained, we have

$$\frac{\partial x}{\partial x_0}\frac{\partial z}{\partial z_0}\Delta x_0\,\Delta z_0-\frac{\partial x}{\partial z_0}\frac{\partial z}{\partial x_0}\Delta x_0\,\Delta z_0=\Delta x_0\,\Delta z_0,$$

where $\Delta x_0\Delta z_0$ is the original area;

$$\therefore\ \frac{\partial x}{\partial x_0}\frac{\partial z}{\partial z_0}-\frac{\partial x}{\partial z_0}\frac{\partial z}{\partial x_0}=1. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is known as Lagrange's equation of continuity.

If the element upon which attention is directed is identified by aid of two parameters a and b, the method of obtaining the equation of continuity needs but a slight modification. Thus, if the parameters a and b are changed to $a+\Delta a$ and b, the point (x, z) will move to the point $\left(x+\frac{\partial x}{\partial a}\Delta a,\ z+\frac{\partial z}{\partial a}\Delta a\right)$, and following through the method of Fig. 127 the volume of the parallelopiped at (x, z) is

$$\frac{\partial x}{\partial a}\frac{\partial z}{\partial b}\Delta a\,\Delta b-\frac{\partial x}{\partial b}\frac{\partial z}{\partial a}\Delta a\,\Delta b;$$

and at some other point at which the element may be found the volume is

$$\frac{\partial x_0}{\partial a}\frac{\partial z_0}{\partial b}\Delta a\,\Delta b-\frac{\partial x_0}{\partial b}\frac{\partial z_0}{\partial a}\Delta a\,\Delta b;$$

and since these volumes are equal,

$$\frac{\partial x}{\partial a}\frac{\partial z}{\partial b}-\frac{\partial x}{\partial b}\frac{\partial z}{\partial a}=\frac{\partial x_0}{\partial a}\frac{\partial z_0}{\partial b}-\frac{\partial x_0}{\partial b}\frac{\partial z_0}{\partial a}. \quad . \quad . \quad . \quad . \quad (15)$$

The parameters a and b may be taken in any convenient manner; if they are made equal respectively to x_0 and z_0, the last equation reduces at once to equation (14), which might have been obtained in this way.

Irrotational Motion. — Let Fig. 128 represent a rectangular element of volume of a rigid substance; if it has an angular velocity α about an axis at the middle point O perpendicular to the section *abcd*, then the resulting velocity of all points on the line *ab* in the direction of that line will be

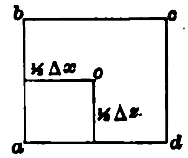


FIG. 128.

$$\Delta v=\tfrac{1}{2}\alpha_1\Delta x. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In like manner a rotation about the same axis with a velocity α_2 will give to points in the line *ad* the velocity

$$-\Delta u=\tfrac{1}{2}\alpha_2\Delta z; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Δu has the negative sign because a right-handed rotation will give a motion toward the left, to points on the line *ad*. Solving equations (16) and (17) for the angular velocities and summing up to get the resultant velocity,

$$\alpha=\alpha_1+\alpha_2=2\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial z}\right) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

at the limit as the volume of the element approaches zero.

If there is no rotation of the element about an axis through its centre, α becomes zero and then

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial z}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

A motion of a fluid in a plane, for which equation (19) can be deduced, is said to be irrotational. If we were dealing with the motion of a fluid in space with three coordinates, there would be three equations like (19) for irrotational motion.

Forces like gravitation and pressure cannot produce rotational motion in a perfect fluid; and conversely, such a fluid which has rotational motion cannot be brought to rest by such forces. This conclusion does not apply to natural fluids which have viscosity.

To prove this proposition consider the forces that act on a rectangular element (Fig. 128). The pressure on a base may vary, but the increment is very small compared with the pressure at a point, and may be neglected in comparison with it; and the resultant of

a uniform pressure will pass through the centre of figure O. The density of the liquid is uniform and the attraction of gravity will consequently pass through O. There remain, therefore, no forces that can produce rotation of the element except tangential forces (or shears) on the faces, and a frictionless liquid cannot be affected by such forces. The conclusion is that gravity and pressure cannot produce rotational motion.

Velocity Potential.—Suppose that there is a function that can be defined by the equations

$$u=-\frac{\partial\phi}{\partial x}, \quad v=-\frac{\partial\phi}{\partial z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where x and z are the rectangular coordinates of a point at which the component velocities parallel to the axes Ox and Oy are u and v. This function ϕ is called the velocity potential, a name that is transferred from the theories of attraction and electrostatics. If there is such a function from which u and v can be derived by differentiation, then these velocities are exact differentials as indicated by the equations (20). And further, if these values are introduced into equation (19), we get

$$\frac{\partial^2\phi}{\partial x\,\partial z}=\frac{\partial^2\phi}{\partial z\,\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which shows that motions imparted to a frictionless fluid by pressure and gravity have a potential. Conversely, if a certain motion of a perfect liquid has not a velocity potential (that is, fails to conform to equation (21)), then that motion cannot be generated by pressure and gravitation.

Great importance is attached by writers on hydrodynamics to the question whether or not a certain motion is irrotational. In the discussion of the theory of waves it will appear that the common or trochoidal theory of waves leads to rotational motion, and it may be that, as these waves are frequently very long and are consequently but little affected by viscosity, this is a just criticism on the theory; there are other objections to the theory which will be pointed out in the proper place. However, that theory will always be a convenient approximation, and it is sufficient for the purposes of the naval architect.

The use of the velocity potential is not necessarily restricted to directions parallel to the coordinate axes. In fact if Δs is a short distance along a given line, and if q is the velocity along that line, then

$$q=-\frac{\partial\phi}{\partial s}. \quad \text{(22)}$$

To show that this is so we may note first that the velocity in a given direction at a point is an absolute quantity which is not dependent on the system of coordinates chosen, and is consequently not changed by changing the axes of coordinates. It is, therefore, sufficient to shift the coordinate axes until one of them (the x axis, for example) is parallel to the given line in order to deduce equation (22) from one of the equations (20).

Suppose that there is a curve which has the same velocity potential at all points; such a line is called an equipotential line, or a line of constant potential. Equation (22) shows that the velocity along a tangent at any point of such a line is zero. Since there is no tangential component to the velocity at a point of an equipotential curve, the velocity at such a point must be along the normal to the curve.

Stream-lines.—The path traced by a particle of fluid in motion is called a stream-line; if the particle remains in a plane, its path is a plane stream-line. Let us consider the motion of a liquid between two parallel planes at an infinitesimal distance apart. The stream-lines of two adjacent particles will mark out an elementary stream, which will have everywhere a rectangular section of infinitesimal area. The liquid flowing in such an elementary stream may be treated as though it were flowing in a tube, for none of the liquid will leave the stream. The law of continuity requires that the volume of liquid per second flowing through such a stream shall be the same at all sections, and consequently the velocity will be inversely proportional to the area of the transverse section of the stream. The flow through any elementary stream is equal to the product of the area by the velocity. The flow through any system of streams is equal to the sum of the flow through all the streams.

A graphical representation of a system of stream-lines can be

made by drawing stream-lines at such intervals that the flow through the intermediate streams shall be equal. In such a graphical representation of stream-lines the lines are drawn at convenient finite intervals; the spacing of the lines will be inversely proportional to the velocity, so that where the lines are near together the velocity is high, and the conditions of the flow will be at once evident to the eye.

Source and Sink.—Flow from a point in a plane will be represented by stream-lines proceeding from that point. Strictly speaking, such a diagram of curves proceeding from a point represents an impossible condition because the area of any stream approaches zero and its velocity approaches infinity as the bounding stream-lines approach a point. To meet this difficulty writers on hydrodynamics consider that the stream-lines terminate in a small circle enclosing the point; it then becomes necessary to suppose that the liquid is furnished in some way to the space inside the circle.

A focus from which stream-lines proceed is called a source; a focus toward which stream-lines converge is called a sink. To get a concrete illustration of a source and a sink we may suppose that water is supplied to a broad shallow tank by a pipe entering through the bottom; in like manner water withdrawn from such a tank by a pipe in the bottom may represent a sink. A diagram representing the flow from a source and toward a sink will be found on page ? .

Stream Function.—In Fig. 129 let CP represent any curve drawn in a plane traversed by stream-lines; corresponding to this curve there will be a cylindrical surface perpendicular to the parallel planes which bound the liquid under consideration. If the velocity normal to the curve at an element Δs is q, then the flow through an elementary area $\Delta s\ \Delta y$ will be

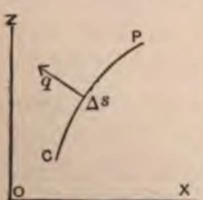


FIG. 129.

$$q\ \Delta s\ \Delta y,$$

where Δy is the distance between the parallel bounding planes. If we desire, we may make the distance between the planes finite, provided that the velocity through all parts of the area

$$y\Delta s$$

is the same. Again, if y be made equal to one foot, the flow through the elementary area having the dimensions Δs along the curve and one foot between the planes will be

$$q\Delta s.$$

The flow under these conditions across the entire line CP will be

$$\psi = \int_C^P q ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This equation holds even though the direction of the velocity is different at different parts of the curve. A flux across the curve from the right toward the left looking from the initial point C toward the point P will be considered to be positive; a flux in the contrary direction will then be negative.

The function ψ is called the stream function of the point P with regard to the point C; it depends only on the location of the points and on the system of stream-lines between P and C, and not on the system of coordinates used, nor on the form of the curve from C to P. To show that the latter statement is true consider the space in Fig. 130 bounded by the curves CDP and CEP and the parallel bounding planes.

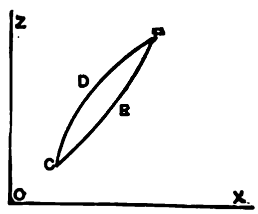


FIG. 130.

If there is no source nor sink within the figure $CDPE$, then the volume of water flowing into that space must be equal to the volume flowing out. Consequently the resultant flux across the boundary CEP must be equal to the resultant flux across CDP. The path from C to P may be shifted at will provided it does not pass over a source or a sink, and may be made up of curves or broken lines.

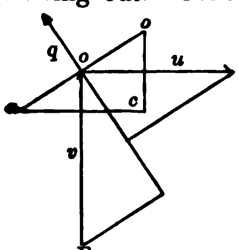


FIG. 131.

In Fig. 130 let abc represent the section of a small triangular figure bounded by a small length Δs of the curve CP in Fig. 131, by two horizontal and vertical lines having the lengths Δx and Δz, and by the two parallel bounding planes. At the point O draw the

two component horizontal and vertical velocities u and v; then if the angle at a is called θ, the resultant normal velocity will be

$$q = -u \sin\theta + v\cos\theta = -u\frac{\Delta z}{\Delta s} + v\frac{\Delta x}{\Delta s}. \quad . \quad . \quad . \quad (24)$$

Substituting this value in equation (23),

$$d\psi = -udz + vdx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

so that

$$u = -\frac{\partial\psi}{\partial z} \quad \text{and} \quad v = \frac{\partial\psi}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If these values are substituted in the equation of continuity (see page 242),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0,$$

we have

$$\frac{\partial^2\psi}{\partial x\,\partial z} = \frac{\partial^2\psi}{\partial z\,\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

which shows that $d\psi$ in equation (25) is an exact differential; that is to say, ψ is a function of the condition of the fluid which does not depend on the manner of passing from one condition to another, as has already been made evident.

Relation of ϕ and ψ.—Comparing equations (20) and (26), it is evident that

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial z} \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -\frac{\partial\psi}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

These equations are important in that they allow us to derive one function from the other.

Addition of Potentials and of Stream Functions.—The velocity potential at a given point may be derived from the equation

$$\phi = \int\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial z}dz\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

or, replacing the partial differential coefficients by their values in equation (20), page 246,

$$\phi = -\int_C^P (udx + vdz), \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where the limits C and P indicate that the integration is to proceed along some path from the initial or reference point to the given point P; the path may, however, be chosen at pleasure.

Suppose that at one time the liquid in the region where the points C and P are located is affected by a motion indicated by the velocities u' and v'; then the velocity potential will be

$$\phi' = -\int_C^P (u'dx + v'dz). \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

If at another time the liquid has a different motion indicated by u'' and v'', the potential will be

$$\phi'' = -\int_C^P (u''dx + v''dz). \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The resultant velocities, if both motions are impressed on the liquid at once, will be

$$u = u' + u'', \quad v = v' + v'', \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

wnich give for the potential in this case

$$\phi = \phi' + \phi'' = -\int_C^P (udx + vdz); \quad . \quad . \quad . \quad (34)$$

that is to say, the velocity potential at a point of a liquid which is affected by the combination of two irrotational motions can be found by adding the potentials due to the individual motions.

Carrying through the same line of reasoning for the stream function ψ gives the same sort of a result; namely, the stream function at a point of a liquid affected by two motions is equal to the sum of the stream functions due to the individual motions. In this case the conclusion is not limited to irrotational motion; but the integration must be along the same path from the reference-point C to

the given point P, or, if the path varies, it must not pass over a source or a sink.

In any case the summation of velocity potentials or of stream functions is independent of the system of coordinates used, and if we choose we may use different coordinates for two motions that are to be combined, noting that the absolute locations of the points C and P must remain unchanged.

Methods of Developing Theories.—In the development of a mathematical theory there are two methods open. Either the conditions determined by observations and experiments may be expressed mathematically and the resulting equations may be combined so as to build up the complete equations which express the theory; or the equations by which a theory can be expressed may be written down, and the conditions involved in such equations may be determined and compared with observations and experiments to see if they are compatible. Sometimes the two methods are combined, especially when there is insufficient information at hand to determine a complete theory, and suggestions for further lines of observation and experiment are desired.

The first method appears to be the more logical and satisfactory; but an attempt to use it when the conditions are not sufficiently known is likely to be plausible rather than convincing. The second method appears artificial, but it has the merit of showing clearly the defects of a theory, if there are any. The choice should be a matter of convenience, since both lead to the same results in the end. Our information concerning the true form of sea-waves of the simplest type is so fragmentary that the second method is preferable.

Trochoidal Waves.—A system of exact equations, expressing a possible form of wave motion when the depth of the water is infinite, was given in 1802 by Gerstner, who deduced them by the first or synthetic method, making the requisite number of reasonable assumptions. This system is sometimes called by his name; more frequently the waves are called trochoidal waves from the form of their profile; the theory advanced by him is the one usually given in books on naval architecture, and is called the common theory for that reason.

With the convention that x is the horizontal and z the vertical coordinates of a particle that can be identified by the parameters a and b, Gerstner's equation may be written

$$\left.\begin{aligned} x&=a+\frac{1}{k}e^{kb}\sin k(a+ct),\\ z&=b-\frac{1}{k}e^{kb}\cos k(a+ct), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x and y are functions of the independent variables a and b; t is the time; e is the base of the Napierian system of logarithms; and k, a and c are constants. It should be noted that x and a are positive to the right and z and b are positive upward. In general the origin of coordinates is entirely above the water, and the parameter of any particle will be essentially negative, but that sign need not be attached to b except for particular purposes.

If these two equations are combined, the resulting equation is transcendental and not in convenient form for use; consequently it is customary to leave them as written.

To show that equations (1) represent a possible hydrodynamic system it is necessary to apply the equations of continuity and equilibrium as developed in the Lagrangian equations in terms of the parameters a and b. Equation (15), page 244, is

$$\frac{\partial x}{\partial a}\cdot\frac{\partial z}{\partial b}-\frac{\partial x}{\partial b}\cdot\frac{\partial z}{\partial a}=\frac{\partial x_0}{\partial a}\cdot\frac{\partial z_0}{\partial b}-\frac{\partial x_0}{\partial b}\cdot\frac{\partial z_0}{\partial a}.$$

The partial differentials in terms of x and y are

$$\frac{\partial x}{\partial a}=1+e^{kb}\cos k\,(a+ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial z}{\partial b}=1-e^{kb}\cos k(a+ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\partial x}{\partial b}=e^{kb}\sin k(a+ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{\partial z}{\partial a}=e^{kb}\sin k(a+ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The left-hand member of the equation of continuity is, consequently,

$$1-e^{2kb}\cos^2 k(a+ct)-e^{2kb}\sin^2 k(a+ct)=1-e^{2kb}. \quad . \quad . \quad (6)$$

The coordinates x and z have the value x_0 and z_0 at the time t_0; consequently the differential coefficients in terms of x_0 and y_0 differ from those written above only in having the subscript zero attached to x, y, and t. It is, therefore, evident that the introduction of the proper differential coefficients into the right-hand member of the equation of continuity will reduce it to the value for the first member; that is, the equation of continuity is satisfied.

The equations of equilibrium (9) and (10), page 241, may be written

$$\frac{\partial}{\partial a}(pw+z) = -\frac{w}{g}\left(\frac{d^2x}{dt^2}\frac{\partial x}{\partial a}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial a}\right), \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{\partial}{\partial b}(p+wz) = -\frac{w}{g}\left(\frac{d^2x}{dt^2}\frac{\partial x}{\partial b}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial b}\right). \quad . \quad . \quad . \quad . \quad (8)$$

For the solution of these equations we need, in addition to the partial differential coefficients of equations (2) to (5), the following differential with regard to time

$$\frac{d^2x}{dt^2} = -c^2ke^{kb}\sin k(a+ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{d^2z}{dt^2} = c^2ke^{kb}\cos k(a+ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Substituting the proper differential coefficients in equations (7) and (8), we get after reduction

$$\frac{\partial}{\partial a}(p+wz) = \frac{w}{g}c^2ke^{kb}\sin k(a+ct) = -\frac{w}{g}\frac{\partial}{\partial a}[c^2e^{kb}\cos k(a+ct)], \quad . \quad (11)$$

$$\begin{aligned}\frac{\partial}{\partial b}(p+wz) &= -\frac{w}{g}c^2ke^{kb}\cos k(a+ct)+\frac{w}{g}c^2ke^{2kb} \\ &= -\frac{w}{g}\frac{\partial}{\partial b}[c^2e^{kb}\cos k(a+ct)]+\frac{w}{g}c^2ke^{2kb} \quad . \quad . \quad (12)\end{aligned}$$

Multiplying equation (11) by da and equation (12) by db and adding, we get the equivalent of

$$d(p+wz) = -\frac{w}{g}d[c^2e^{kb}\cos k(a+ct)]+\frac{w}{g}c^2ke^{2kb}db \quad . \quad . \quad (13)$$

Integration gives

$$p+wz=-\frac{w}{g}c^2e^{kb}\cos(a+ct)+\frac{1}{2}\frac{w}{g}c^2e^{2kb}+\text{const.}; \quad . \quad (14)$$

and replacing z by its value, equation (1),

$$p=-wb+\left(\frac{w}{k}-\frac{wc^2}{g}\right)e^{kb}\cos k(a+ct)+\frac{1}{2}\frac{w}{g}c^2e^{2kb}+\text{const.} \quad (15)$$

To apply equation (15) to a particle at the free water surface we may note that the pressure p is there the pressure of the atmosphere which is sensibly constant; and that b will have a particular value assigned to it. Consequently every term of equation (15) will under the conditions be constant except the one containing t. If we make

$$c^2=\frac{g}{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

that term will disappear. This is the necessary and sufficient condition in order that equation (1) shall conform to the equation of continuity.

Since both the equations for equilibrium and for continuity are satisfied by Gerstner's equations, the system of waves represented by them is hydrodynamically possible; the motion is rotational, as will be seen by applying as the test equation (19)

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial z}=0.$$

Values for the component velocities can be obtained by differentiating equations (1) as follows:

$$u=\frac{dx}{dt}=ce^{kb}\cos k(a+ct), \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$v=\frac{dz}{dt}=ce^{kb}\sin k(a+ct). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The values of the partial differential coefficients of these component velocities may be obtained as follows:

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial a}\frac{\partial a}{\partial z}+\frac{\partial u}{\partial b}\frac{\partial b}{\partial z}=\frac{\partial u}{\partial a}\div\frac{\partial z}{\partial a}+\frac{\partial u}{\partial b}\div\frac{\partial z}{\partial b}, \quad . \quad . \quad . \quad (19)$$

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial v}{\partial b}\frac{\partial b}{\partial x}=\frac{\partial v}{\partial a}\div\frac{\partial x}{\partial a}+\frac{\partial v}{\partial b}\div\frac{\partial x}{\partial b}. \quad . \quad . \quad (20)$$

The partial differential coefficients of u and v with regard to a and b may be derived from equations (17) and (18), and the other differential coefficients may be taken from equations (2) to (5). Substitution and reduction gives

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial z}=2kc\left[1-\frac{e^{2kb}\cos^2 k(a+ct)}{1-e^{2kb}\cos^2 k(a+ct)}\right]. \quad . \quad . \quad . \quad (21)$$

Properties of Trochoidal Waves.—The properties of a system of trochoidal waves may be readily inferred from equations (1) which are repeated for convenience.

$$x=a+\frac{1}{k}e^{kb}\sin k(a+ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$z=b-\frac{1}{k}e^{kb}\cos k(a+ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The point which has the coordinates x and z is at the horizontal distance $x-a$ from a point which has the parameters a and b for its coordinates; and the vertical distance is $z-b$. The absolute distance between the two points is

$$r=\{(x-a)^2+(z-b)^2\}^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

or reducing by aid of equations (22) and (23),

$$r=\frac{1}{k}e^{kb}\{\sin^2 k(a+ct)+\cos^2 k(a+ct)\}^{\frac{1}{2}}.$$

$$\therefore\ r=\frac{1}{k}e^{kb}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From this it is evident that r is constant for any given particle, and that consequently each particle revolves in a circle about a centre, which has the parameters a and b for its coordinates. The mean depth of a particle is the depth of its centre of revolution; and from equation (25) all particles having the same mean depth b have the same diameter of orbit. There is no drift or current produced in the water by a system of trochoidal waves.

The angle $k(a+ct)$ in equations (23) and (24) increases directly with the time; consequently the angular velocity is kc for all particles

in their orbits. A complete revolution is represented by the angle 2π; consequently the time of a revolution is

$$T = \frac{2\pi}{kc}. \quad \text{. . . . .} \quad (26)$$

If x in equation (22) is made equal to a, the second term of the second member becomes equal to zero, so that

$$k(a+ct_1)=0, \text{ or } k(a+ct_2)=\pi. \quad \text{. . . . .} \quad (27)$$

The first value makes

$$z_1 = b - \frac{1}{k}e^{kb}, \quad \text{. . . . .} \quad (28)$$

and the second value gives

$$z_2 = b + \frac{1}{k}e^{kb}. \quad \text{. . . . .} \quad (29)$$

The second condition corresponds to a location of the particle at the top of its orbit directly over the centre of orbit; a particle of water at the free surface will then be at the crest of a wave. The first condition places a particle directly under its centre of orbit; at the free surface this is the hollow of a wave.

All particles which have the same parameter a will come to the top of their orbits at the same time, and in general will be at the same phase of oscillation. As the time varies the location of the particle which is at the top of its orbit will change; that is, the crest of the wave will move forward. To investigate this motion write

$$k(a+ct)=0,$$

and differentiate a with regard to t, which gives

$$\frac{da}{dt} = -c. \quad \text{. . . . .} \quad (30)$$

This shows that the velocity of the crest of the wave is c feet per second toward the left.

After the time T the crest of the wave will have advanced the distance

$$L = cT, \quad \ldots \ldots \ldots \quad (31)$$

and the original particle will have completed a revolution and be again at the crest of a wave; consequently L is the length of a wave from crest to crest. The length of the wave may also be reckoned from hollow to hollow, or in general from a given phase to the same phase of the next wave. For, since T is the same for all particles, the length of an oscillation measured between particles having the same diameter of orbit and the same phase is the same.

Resuming equation (15) with the condition imposed by equation (16), we have

$$p = -wb + \frac{1}{2}\frac{w}{g}c^2e^{2kb} + \text{const.}, \quad \ldots \ldots \quad (32)$$

which shows that the pressure is the same at all particles having the same parameter b. It has already been seen from equation (25) that all such particles have the same diameter of orbit. A surface formed of all particles having the same parameter b is called a wave surface; since the pressure on any surface is uniform, it is under the same conditions as it would be if it were the free surface of the water exposed to the atmosphere.

The vertical profile of a wave surface perpendicular to the line of a wave crest is a trochoid. This may be conveniently shown by making the time equal to zero in equations (22) and (23) and introducing r from equation (25), giving

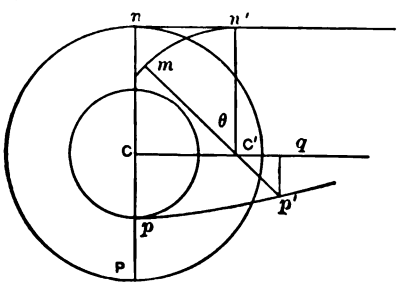


FIG. 132.

$$x = a + r \sin ka \quad . \quad (33)$$

$$z = b - r \cos ka. \quad . \quad (34)$$

If the length of a trochoid is L, then the diameter of the rolling circle that may be used to describe the trochoid is

$$R = \frac{L}{2\pi}. \quad \ldots \ldots \ldots \quad (35)$$

Replacing L by its value by equation (31) and introducing the value of T from equation (26) and reducing,

$$R=\frac{1}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If this value of k is introduced into equations (33) and (34), they become

$$x=a+r\sin\frac{a}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$z=b-r\cos\frac{a}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Beginning at a hollow where a is zero, the parameters of the particle p are $a=0$ and b, which locate the centre C (Fig. 132) of its orbit. A particle having the parameters a and b will have its centre of orbit at C' and will have the coordinates given by the equations just written. About C draw a rolling circle with the radius R and let it roll the distance $nn'=a$ along the horizontal line nn'; its centre will be at C', and it will turn through the angle $\theta=a\div R$; consequently the describing-point will move from p to p', which latter will evidently have its coordinates given by equations (37) and (38), which shows that the describing-point passes to the particle p', and that the wave profile is a trochoid.

The geometrical limit for trochoidal waves is taken to be the cycloid. To investigate this case let b be made zero in equation (25), giving

$$r_m=\frac{1}{k}=R, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

by equation (36). This condition makes O in Fig. 132 the origin of coordinates, and the tracing-point passes to P on the circumference of the describing circle. The vertical acceleration due to the motion of the wave as given by equation (10) now becomes

$$\frac{d^2z}{dt^2}=c^2k\cos kct, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

because the parameters a and b are both zero for this case. Replacing the coefficient of the cosine by its value from equation (16), the acceleration becomes

$$\frac{d^2z}{dt^2}=g\cos kct. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

When the particle is at P (Fig. 132), i.e., when it is at the hollow of a wave, the time is taken to be zero and the acceleration due to the wave motion is equal to g, the acceleration due to gravity; so that the total acceleration is $2g$, or twice that due to gravity. The time for an entire oscillation is given by equation (26), from which it is evident that the time of a half-oscillation, during which a particle passes from the hollow to the crest, is

$$\tfrac{1}{2}T=\frac{\pi}{kc}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Consequently the acceleration at the crest of a cycloidal wave is

$$\frac{d^2z}{dt^2}=g\cos\pi=-g;$$

that is, the acceleration due to gravity is just equal to the required acceleration. If the radius of the orbit were greater than R, gravity could not produce the required acceleration at the crest of a cycloidal wave, and it would break. In reality sea-waves break at heights much less than that indicated by this discussion, as will appear in the theory of waves which follows this one.

Length, Time, and Speed of Waves.—Solving equation (31) for T and multiplying the resulting equation by equation (26) gives

$$T^2=\frac{2\pi L}{kc^2};$$

replacing c^2 by its value in equation (16) gives

$$T=\sqrt{\frac{2\pi L}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

This equation, together with the following,

$$c=\frac{L}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

which comes directly from equation (31), gives the means of calculating the most important relations of waves. The following table is computed by them:

LENGTH, TIME, AND SPEED OF WAVES.

Length, Feet.	Time, Seconds.	Speed, Feet per Second.	Speed in Knots.
50	3.10	16.0	9.5
100	4.40	22.7	13.4
200	6.20	32.0	19.0
400	8.80	45.3	26.8
600	10.80	55.0	32.8
800	12.50	64.1	37.9
1000	14.0	71.6	42.4
2000	19.8	101.5	60.0

Law of Depth.—From equation (25),

$$r=\frac{1}{k}e^{kb},$$

it can be shown that the radius of the orbit of a particle decreases rapidly with its depth from the surface. For if b_0 is the parameter of a particle at the free surface of the water and b is the parameter of an under-water particle, then

$$\frac{r}{r_0}=e^{-k(b_0-b)}=e^{-kD}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where D is the mean depth of the particle below the surface. But from equations (26) and (31)

$$k=\frac{2\pi}{cT}=\frac{2\pi}{L}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

therefore

$$\frac{r}{r_0}=e^{-\frac{2\pi D}{L}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The following table was computed by aid of equation (47):

DECREASE OF WAVE DISTURBANCE WITH DEPTH

Ratio of Depth to Length, $D \div L$	Ratio of Orbits of Particles, $r \div r_0$
0.01	0.9391
0.02	0.8819
0.03	0.8283
0.04	0.7778
0.05	0.7304
0.1	0.5335
0.2	0.2846
0.3	0.1518
0.5	0.0432
1.0	0.00187
2.0	0.0000035

Pressure on a Wave Surface.—It has been shown on page 258 that the pressure on a wave surface, made up of particles having the same parameter b, is uniform and equal to

$$p = -wb + \frac{1}{2}\frac{w}{g}c^2 e^{2kb} + \text{const.};$$

replacing c^2 by its value from equation (16), and e^{kb} by its value from equation (25), and k by its value from equation (46), we have

$$p = -wb + \frac{\pi w}{L}r^2 + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

A particle at the free surface with the parameter b_0 will be subjected to the pressure p_0, which is, of course, the pressure of the atmosphere. The increase of pressure due to decrease of parameter from b_0 to b will be

$$p - p_0 = w(b_0 - b) - \frac{\pi w}{L}(r_0^2 - r^2) \quad . \quad . \quad . \quad . \quad . \quad (49)$$

This increase of pressure in pounds per square foot may be reduced to equivalent hydraulic head by dividing by w, the weight of one cubic foot of water in pounds; calling the head H, we have

$$H = b_0 - b - \frac{\pi}{L}(r_0^2 - r^2). \quad . \quad . \quad . \quad . \quad . \quad (50)$$

If we choose we may represent the change of parameters by D, which will represent the distance of a particle below the free surface, measured from the centre of the orbit of the given particle to the centre of the orbit of a particle at the free surface, giving

$$H = D - \frac{\pi}{L}(r_0^2 - r^2); \quad \ldots \ldots \quad (51)$$

and again r may be replaced by its value in equation (47), giving

$$H = D - \frac{\pi}{L} r_0^2 (1 - e^{\frac{-2\pi D}{L}}) \ldots \ldots \quad (52)$$

The equations (51) and (52) make it possible to compute the equivalent hydrostatic pressure at any wave surface at the depth D below the free surface; if the equation (51) is selected, the value of r may be taken from the table on the opposite page.

Energy of a Wave.—Water which is affected by a trochoidal motion has in it both kinetic and potential energy; the presence of kinetic energy is at once evident since every particle is revolving in an orbit with a known velocity; but to show that the water has potential energy it is necessary to show that a particle is higher when affected by such a wave motion than when the water is at rest. For this purpose let us apply equation (52) to a particle at a depth which is very large compared with L, so that approximately

$$H = D - \frac{\pi}{L} r_0^2 = b_0 - b - \frac{\pi}{L} r_0^2. \quad \ldots \ldots \quad (53)$$

Let us consider that the wave motion is insensible at that depth and that the hydrostatic pressure is the same as when there is no wave motion. Assume that H and b in equation (53) remain unchanged and that r_0 decreases; as r_0 approaches zero, b_0 will approach a limiting value b_0', so that at the limit

$$H = b_0' - b; \quad \ldots \ldots \ldots \quad (54)$$

that is, with no wave motion the hydrostatic head at a particle is equal to its depth below the free surface. Subtracting equation (53)

from equation (54) and transposing,

$$b_0 - b_0' = \frac{\pi}{L} r_0^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

which shows that a particle at the free surface of a trochoidal wave is $\pi r_0^2 \div L$ feet higher than when at rest. This discussion has been restricted to the free water surface, but that restriction need not be retained, for, beginning with equation (49), the pressure p_0 may be taken as that at any wave surface and p as the pressure at a much greater depth, and the conclusion will then be identically the same.

Consider a thin layer of water at the depth H below the free surface when the water is at rest; every particle of this surface will rise a distance $\pi r^2 \div L$ when it is affected by a trochoidal wave motion, and every partical of the free water surface will rise the distance $\pi r_0^2 \div L$. The depth of a particle of the wave surface which was originally at the depth H, will now be

$$D = H + \frac{\pi}{L}(r_0^2 - r^2),$$

and consequently

$$H = b_0 - b - \frac{\pi}{L}(r_0^2 - r^2),$$

as in equation (50). Therefore the thickness of the layer was originally

$$-dH = -db + \frac{2\pi r}{L} dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

Here dH has the negative sign because H decreases as b increases. From equation (25),

$$kb = \log_{\hat{e}} kr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

$$\therefore \; db = \frac{1}{kr} dr = \frac{L}{2\pi r} dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (58)$$

the last transformation being by aid of equation (46). This gives for the positive thickness of the undisturbed layer

$$dH = \left(\frac{L}{2\pi r} - \frac{2\pi r}{L} \right) dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (59)$$

The potential energy due to raising such a layer of water one foot wide and L feet long the distance $\pi r^2 \div L$ will be

$$dE_p = wL \cdot \frac{\pi r^2}{L}\left(\frac{L}{2\pi r} - \frac{2\pi r}{L}\right)dr. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

This expression is to be integrated from the surface where $r_0 = r$ to an infinite depth where $r = 0$; with these limits the above equation gives

$$E_p = wL\int_0^{r_0}\left(\frac{r}{2} - \frac{2\pi^2 r^3}{L^2}\right)dr. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$\therefore E_p = \tfrac{1}{4}wLr_0^2\left(1 - \frac{2\pi^2 r_0^2}{L^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The angular velocity of a particle in its orbit (see page 256) is kc, so that the linear velocity is rkc. The square of the linear velocity is, therefore, by aid of equation (16) and equation (46), equal to

$$\frac{2\pi g r^2}{L};$$

and consequently the kinetic energy of a layer of water which originally had a thickness dH, a width of one foot, and a length of L feet is

$$dE_k = \frac{1}{2}\,\frac{wL}{g}\left(\frac{L}{2\pi r} - \frac{2\pi r}{L}\right)\frac{2\pi g r^2}{L}dr. \quad . \quad . \quad . \quad . \quad (63)$$

$$\therefore E_k = wL\int_0^{r_0}\left(\frac{r}{2} - \frac{2\pi^2 r^3}{L^2}\right)dr. \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$\therefore E_k = \tfrac{1}{4}wLr_0^2\left(1 - \frac{2\pi^2 r_0^2}{L^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

The potential and kinetic energies are equal and the total energy is

$$E = \tfrac{1}{2}wLr_0^2\left(1 - \frac{2\pi^2 r_0^2}{L^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

or, replacing r_0 by $\frac{1}{2}h$, where h is the height of the wave from hollow to crest at the surface,

$$E=\frac{1}{8}wLh^2\left(1-\frac{\pi^2h^2}{2L^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Here h is the height of the wave in feet, L is the length in feet, and w is 62.4 pounds for fresh water or 64 pounds for sea-water. If the wave is not high, the second term in the parenthesis can be omitted.

Irrotational Waves.—A theory of waves having irrotational motion has been developed by Sir G. Stokes which gives a form similar to but differing appreciably from the trochoid. The most convenient way of treating this problem is to suppose that the water affected by the wave motion had originally a uniform velocity upon which has been superposed an oscillation giving rise to a series of waves with crests that have a velocity equal and contrary to the original uniform velocity. Such a combination of motions will give rise to a series of stationary waves through which the water flows along definite steam-lines; an obstacle in the bed of a uniform stream flowing in a canal will give rise to such a system of waves.

A system of stationary waves in water of infinite depth may be represented by the following equations for velocity potential, and stream functions:

$$\frac{\phi}{c}=-x+\beta e^{kz}\sin kx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\psi}{c}=-z+\beta e^{kz}\cos kx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where x (positive to the right) and z (positive upwards) are the coordinates referred to arbitrary horizontal and vertical axes; β and k are arbitrary constants whose values are to be determined, and c is the original uniform velocity of the water; e is the base of the Napierian system of logarithms.

The existence of a velocity potential infers irrotational motion, for the component velocities

$$u=-\frac{\partial\phi}{\partial x} \text{ and } v=-\frac{\partial\phi}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

when introduced into equation (19), page 245, give at once

$$\frac{d^2\phi}{dx\,dz}=\frac{d^2\phi}{\partial z\,\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To show that the stream function agrees with the velocity potential it is sufficient to note that

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial z}=-c(1-k\beta e^{kz}\cos kx)=-u, \quad . \quad . \quad . \quad (5)$$

$$\frac{\partial\phi}{\partial z}=-\frac{\partial\psi}{\partial x}=ck\beta e^{kz}\sin kx=-v, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

as required by the equations (28), page 250.

The values of u and v just determined when introduced into the equation (12), page 242, give

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=-ck^2\beta e^{kz}\sin kx+ck^2\beta e^{kz}\sin kx=0, \quad . \quad . \quad (7)$$

which shows that the motion conforms to the condition for continuity.

To investigate the conditions for equilibrium equations (4), page 240, may be written

$$\frac{\partial p}{\partial x}=-\frac{w}{g}\frac{du}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{\partial p}{\partial z}=-\frac{w}{g}\left(g+\frac{dv}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From these equations a convenient form for the present purpose can be obtained by a series of transformations. In the first place the complete differentials of the component velocities can be expressed in terms of partial differential coefficients giving

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}\frac{dx}{dt}+\frac{\partial v}{\partial z}\frac{dz}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

But

$$\frac{dx}{dt}=u, \quad \frac{dz}{dt}=v;$$

also

$$\frac{\partial u}{\partial t}=-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right), \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{\partial v}{\partial t}=-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z}\right)=-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right), \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and again from equation (19), page 245

$$\frac{\partial v}{\partial x}=\frac{\partial u}{\partial z};$$

so that equations (10) and (11) may be written

$$\frac{du}{dt}=-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{dv}{dt}=-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)+u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If these values of the partial differential coefficients are introduced into equations (8) and (9), and if further we multiply the resulting equations one by dx and the other by dz and add, we shall get

$$\frac{g}{w}dp=d\left(\frac{\partial\phi}{\partial t}\right)-gdz-udu-vdv, \quad . \quad . \quad . \quad . \quad (16)$$

and on integration

$$\frac{g}{w}p=\text{const.}+\frac{\partial\phi}{\partial t}-gz-\tfrac{1}{2}(u^2+v^2). \quad . \quad . \quad . \quad . \quad (17)$$

Thus as a result of the transformations the equation of equilibrium is given in an integral form with the exception of the differential coefficient of the velocity potential with regard to time; and as equation (1) does not involve time, that differential coefficient is zero for the present discussion.

Introducing values of the velocities u and v from equations (5) and (6), the above equation of pressure becomes

$$\frac{g}{w}p = \text{const.} - gz - \tfrac{1}{2}c^2\{1 - 2k\beta e^{kz}\cos kx + k^2\beta^2 e^{2kz}\}. \quad . \quad (18)$$

The equation to a stream-line is obtained by making ψ constant in equation (2), which gives

$$\beta e^{kz}\cos kx = z + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Introducing this value into equation (18) and uniting all the constant terms into one general constant, we have

$$\frac{g}{w}p = \text{const.} + (c^2k - g)z - \tfrac{1}{2}k^2c^2\beta^2e^{2kz}. \quad . \quad . \quad . \quad (20)$$

In order that this equation may conform to the condition that the pressure at the free surface shall be constant and equal to the pressure of the atmosphere, it is necessary now to show that the terms on the right-hand side of this equation are constants or that they can be made to disappear.

Expanding the exponential term by Maclaurin's theorem,

$$e^{2kz} = 1 + 2kz + 2k^2z^2 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If it can be assumed that quantities containing higher powers of k than the cube may be omitted, equation (20) may be reduced to

$$\frac{g}{w}p = \text{const.} + (c^2k - g - k^3c^2\beta^2)z - \tfrac{1}{2}k^2c^2\beta^2,$$

or, uniting the last term with the general constant,

$$\frac{g}{w}p = \text{const.} + (c^2k - g - k^3c^2\beta^2)z. \quad . \quad . \quad . \quad . \quad (22)$$

If it be assumed that

$$c^2 = \frac{g}{k} + k^2c^2\beta^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

then the parenthesis of equation (22) disappears; consequently that is the necessary condition that the pressure along a stream-line shall be constant; and in particular that the pressure at the free surface shall be constant and equal to that of the atmosphere. It will

appear later that k and β are both small, and for an approximation we may make

$$c^2=\frac{g}{k},$$

and apply that approximation in the second member of equation (23), giving

$$c^2=\frac{g}{k}(1+k^2\beta^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This equation gives an approximate value for the uniform velocity of flow of the water on which the wave motion is superposed.

The further analytical discussion given by Stokes is somewhat intricate, especially if his approximations are carried to a sufficient degree to show the difference between this wave and the trochoidal wave. It may be enough to state his conclusion that if quantities involving the cube and higher powers of β are omitted, then the equation for the wave profile can be shown to coincide with the equation to a trochoid which has the radius of the tracing-point

$$r=\beta(1+k^2\beta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

the centre of the rolling circle being at the distance

$$\frac{\pi r^2}{L}$$

above an arbitrary axis of abscissæ.

A comparison of this irrotational wave with the trochoidal wave can readily be made graphically; and the form of the highest wave of the system can be determined by trial, depending on an independent proof by Stokes that the sharpest crest of an irrotational wave will have an angle of 120°. The proof is as follows: let attention be confined to the water in the neighborhood of a crest, and assume that crest to be bounded by the tangent lines RO and OS, Fig. 133. The stream-line may be assumed to be represented by the equation

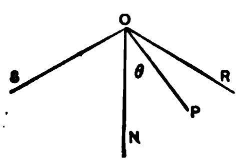


FIG. 133.

$$\psi=-Cr^m \cos m\theta, \quad . \quad . \quad . \quad (26)$$

where the coordinates of a point P are r and θ referred to the pole

O and the axis ON, drawn vertically downward; and C and m are constants. To show that this represents irrotational motion we may apply equation (21), page 246,

$$\frac{\partial^2\phi}{\partial x\,\partial z}=\frac{\partial\phi^2}{\partial z\,\partial x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in which we may introduce the proper partial differentials by aid of equation (28), page 250:

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial z} \quad \text{and} \quad \frac{\partial\phi}{\partial z}=-\frac{\partial\psi}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Now it appears from the nature of the stream function that it is not affected by the system of coordinates used, and consequently we may use any system we find convenient and shift the coordinates as we choose. Let us take O for the origin of rectangular coordinates and OP for the axis of x, then

$$\Delta x=\Delta r \quad \text{and} \quad \Delta z=r\Delta\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

consequently

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial z}=\frac{m}{r}Cr^m \sin m\theta. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\therefore \frac{\partial^2\phi}{\partial x\,\partial z}=\frac{m^2}{r^2}Cr^m \cos m\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and

$$\frac{\partial\phi}{\partial z}=-\frac{\partial\psi}{\partial x}=mCr^{m-1} \cos m\theta. \quad . \quad . \quad . \quad (32)$$

$$\therefore \frac{\partial^2\phi}{\partial z\,\partial x}=m^2Cr^{m-2} \cos m\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

which shows that the conditions for equation (27) are fulfilled. But the same operation can be applied to all points in the region considered; which is therefore a region of irrotational motion.

The velocity of a particle at P along the radius vector is, by equation (30),

$$u=-\frac{\partial\phi}{\partial x}=-mCr^{m-1} \sin m\theta, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and equation (32) gives for the velocity at right angles to the radius vector

$$v = -\frac{\partial \phi}{\partial z} = -mCr^{m-1} \cos m\theta, \quad . \quad . \quad . \quad . \quad . \quad (35)$$

so that the resultant velocity is

$$q = (u^2 + v^2)^{\frac{1}{2}} = mCr^{m-1}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In the angle *NOR*, where the water is moving toward the crest, the velocity has the negative sign, and in the angle *NOS*, where it is moving away, it has the positive sign. At the origin where r is zero the velocity is zero.

If it can be considered that all the water near the crest is nearly at rest, then the velocity at any point may be found approximately by the hydrostatic formula

$$q = \sqrt{2gh},$$

where h is the hydrostatic head. Replacing h by $r \cos \theta$, the velocity becomes

$$q = \sqrt{2gr \cos \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Combining equations (36) and (37),

$$m^2C^2r^{2m-3} = 2g \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The free water-surface is a stream-line, and ψ is a constant and may be made equal to zero, so that equation (26) gives in this case

$$\cos m\theta_0 = 0 \quad \text{and} \quad \theta_0 = \frac{\pi}{2m}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

which makes θ a constant; and in order that equation (38) may be true for this case the remaining variable r must be made to disappear also. This will happen if its exponent is equal to zero, that is, if

$$m = \tfrac{3}{2}.$$

Applying this value of m to equation (39) makes the half-angle at the crest equal to 60° and consequently the whole angle *ROS* is 120°.

The exact equation to the wave profile can be obtained by making $\psi = 0$ in equation (2), giving

$$z = \beta e^{kz} \cos kx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

It is evident that there is a crest at the origin where $x = 0$, because the equation is then satisfied by positive values of z. On the other hand the equation is satisfied by negative values of z at

$$x_1 = \tfrac{1}{2}L = \pi \div k,$$

and there is therefore a hollow there. This gives at once for the length of a wave

$$L = \frac{2\pi}{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

as for trochoidal waves.

The curve crosses the axis of x where z is zero; this requires that $\cos kx$ shall be equal to zero; consequently

$$x_2 = \frac{\pi}{k} = \tfrac{1}{4}L \quad \text{and} \quad x_3 = \frac{3\pi}{k} = \tfrac{3}{4}L$$

are points where the curve crosses the axis of x.

To find the ordinate of the curve at the origin, make $x = 0$ in equation (40), whence

$$z = \beta e^{kz}; \quad \therefore \log_e \frac{z}{\beta} = kz.$$

$$\therefore \frac{z}{L} = \frac{1}{2\pi M} \log_{10} \frac{z}{\beta} = 0.3664 \log_{10} \frac{z}{\beta}; \quad . \quad . \quad . \quad . \quad (42)$$

the last transformation being by aid of equation (41), together with the transformation to common logarithms whose modulus is M. The ordinate of the curve at a hollow of the wave where $x = \frac{1}{2}L$ can, in a similar way, be shown to be found by the equation

$$\frac{z}{L} = -0.3664 \log_{10} \frac{\beta}{z}. \quad . \quad \quad . \quad . \quad . \quad . \quad (43)$$

Other points in the curve can be found by solving equation (40) for x; thus,

$$x=\frac{1}{k}\cos^{-1}\frac{z}{\beta e^{kz}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

It has been found by trial that if β is taken equal to $0.058446L$, the wave profile shown by Fig. 134 has a sharp crest with an angle of about 120°. The height of the wave from hollow to crest is nearly one-fifth of the length. If a medial line is drawn by trial that will divide the wave so that the area of crest (above the line)

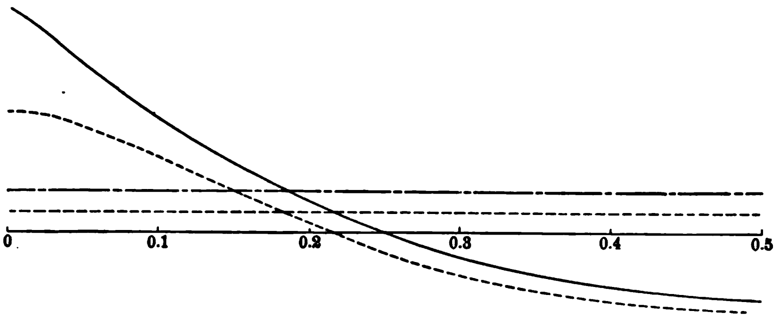


FIG. 134.

and hollow (below the line) are equal, it will be about .028 of the wave length above the arbitrary axis. A trochoid drawn by the directions on page 270 will have the centre of its rolling circle $0.0138L$ above the arbitrary axis, and the height of such a trochoidal wave will be less than one-eighth of the length. This trochoid is drawn on Fig. 134 to exhibit the discrepancy. The discrepancy diminished

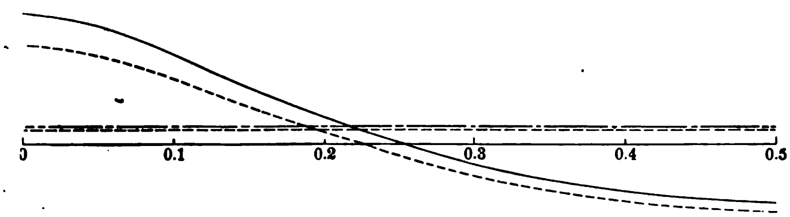


FIG. 135.

rapidly with the height of the wave, as seen from Fig. 135, which gives the profile of a wave with $\beta=\frac{1}{10}L$. Here the height of the

irrotational wave is $0.125L$, and of the corresponding trochoidal wave is $0.11L$. The two forms of wave are practically coincident when the height is one-twentieth (or less) of the length, and for such waves the properties of the trochoidal waves may be assumed for the irrotational waves without much error.

Michell * gives another discussion of irrotational waves with the conclusion that a wave which has a sharp crest at the angle of 120° has a height of $0.142L$, and that its velocity is 1.2 times that of a very low wave.

In the statement of this method of determining a system of irrotational waves it was assumed that the water had an original uniform motion with the velocity c, and that the wave motion was superposed on this uniform stream. It is clear that the same sort of relative motion will be found for waves advancing through still water with a velocity $-c$, that is, toward the left.

The discussion given above for the form of the crest of a wave having the maximum height shows that a particle of water at a crest of such a maximum wave will have a velocity equal to that of the wave and in the same direction. The theory of trochoidal waves also shows that for the maximum height a particle at the crest of a wave has the same velocity as the wave itself; because at the maximum height the radius of the orbit is $R=L\div 2\pi$, and the angular velocity is $k=2\pi c\div L$, and consequently the linear velocity is equal to c. It should be noted that the theoretical maximum height of a trochoidal wave is nearly twice that for an irrotational wave, being equal to $2R=L\div\pi$.

The appropriate equation for speed on page 270

$$c^2=\frac{g}{k}(1+k^2\beta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

shows that the velocity depends on the height of the wave. For a very small height the velocity is equal to

$$c=\sqrt{\frac{g}{k}}=\sqrt{\frac{gL}{2\pi}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

the same as for trochoidal waves, equation (16), page 255.

* Phil. Mag., Nov. 1893.

Equation (45) applied to the wave of maximum height for which β is $0.058446L$ gives a velocity which is 1.13 times that given by equation (46). On the other hand, if β is $\frac{1}{10}L$, the velocity by equation (45) is only 1.02 times that by equation (46).

Irrotational waves of the type under discussion have a drift or current in the direction in which the waves are running. To make this evident we may compute the velocity in that direction by aid of equation (5), page 267,

$$u = c(1 - k\beta e^{kz} \cos kx),$$

for a series of values of x, taking z from the contour of the wave in Fig. 134, with the following results:

$x=0$	$0.05L$	$0.1L$	$0.15L$	$0.2L$	$0.25L$
$u=0$	$0.291c$	$0.520c$	$0.711c$	$0.871c$	$1.00c$

$x=0.3L$	$0.35L$	$0.40L$	$0.45L$	$0.5L$
$u=1.102c$	$1.180c$	$1.235c$	$1.268c$	$1.278c$

The arithmetical mean of these values is $0.860c$; this is the relative velocity of the surface-water compared with the velocity of water at a great depth; and if we assume that the water at that depth is at rest and that consequently the crest of the waves has a velocity $-c$ feet per second, the mean velocity of the surface-water will be $-1.14c$; this last quantity is consequently the mean velocity or drift of the surface-water for waves having the maximum height. This drift decreases rapidly with the depth and approaches zero at a great depth, where the value of z is negative and where consequently e^{kz} in the equation for the velocity approaches zero.

As an example it appears that the speed of an irrotational wave of the maximum height and 100 feet long is 22.6 feet per second or 13.4 knots per hour. The drift of the maximum wave 100 feet long is 3.2 feet per second or 1.8 knots per hour.

Waves which have a small height compared with the length have already been said to be approximately trochoidal; if the height is very small, the contour may be considered to be a curve of sines. For at the surface, as already seen, equation (2) may be written

$$z = \beta e^{kz} \cos kx,$$

and e^{kz} may be replaced by unity, while β may be replaced by r_0. At the same time k may be replaced by its value $2\pi \div L$, so that

$$z = r_0 \cos kx = r_0 \cos \frac{2\pi x}{L}. \quad . \quad . \quad . \quad . \quad . \quad (47)$$

An equivalent to this equation can be obtained from the equations (38) and (39) for trochoidal waves if b is made equal to zero and a is made equal to x, because $R = 2\pi \div L$. This is, in effect, equivalent to taking account of vertical displacements of the revolving particle of water and ignoring the horizontal displacements; these displacements are, in fact, of the same order of magnitude, but for certain purposes the vertical displacements are conveniently compared with the height of the wave, while the horizontal displacements are compared with the length. Another way of looking at the matter is simply to note that for small heights both the trochoidal wave and the exact irrotational wave have contours that are closely represented by a curve of sines.

Observations on Waves.—The only observations that have been made on waves at sea are on the length, height, and speed, and even these observations are difficult and unsatisfactory, especially as, under the most favorable conditions, both the length and height of a system of sea waves vary appreciably.

It is evident that the influence of viscosity on large waves is small, since they persist for a long time after the wind ceases and run for long distances with little change; and yet viscosity has an appreciable effect, since the waves finally come to rest. The length is more persistent than the height; indeed, it is not easy to see how the length is affected by viscosity.

Observations are reported by Lieut. A. Pâris* on the same system of waves at two places 350 miles apart, as follows:

	First Place.	Second Place.
Speed, meters per second	15.3	15
Length "	143	145
Height "	4.5	2.25

* Observations sur l'état de la mer. Revue Maritime, Vol. XXXI, p. 111.

The second observation gives a very good agreement with the theory when the speed is computed by equation (46), page 275; the first shows a greater speed than is indicated by that equation; in both cases the height is too small to introduce any appreciable difference between the results by equations (45) and (46).

Waves have been observed with periods of 22 to 24 seconds and lengths of 2500 to 3000 feet. The greatest measured height of waves is 45 feet; waves 40 feet high are rare, and 30 feet is an uncommon height. The ratio of height to length tends to diminish as the length increases. The common value of the ratio of height to length is from 0.05 to 0.1; very long waves have about 0.02 for that ratio.

The following table gives the dimensions and other interesting information concerning waves subject to regular winds in several well-known seas, quoted from the memoir by Lieut. Pâris already mentioned:

PROPERTIES OF WAVES IN VARIOUS SEAS.

Region.	Depth of Sea.	Velocity of Wind, V, Knots per H.	Period of Waves in Seconds.	Length, Feet, L.	Height, Feet, $h = 2r$.	Speed, C. Feet per Sec.	Knots per Hour	Ratio $\frac{h}{L}$.	Ratio $\frac{V}{C}$.
Atlantic trade winds	12800	9	5.8	213	6.2	36.8	21.8	.0293	0.43
South Atlantic region of west winds	14230	26	9.5	436	14.1	45.9	27.2	.0323	0.96
Indian Ocean trade winds	13700	12	7.6	315	9.2	41.3	24.5	.0292	0.52
Indian Ocean region of east winds	12000	33	7.6	374	17.4	49.2	29.1	.0465	1.16
Pacific (west)	14750	14	8.2	335	10.2	40.7	24.1	.0304	0.68
Japan and China seas	3160	28	6.9	259	10.5	37.4	22.1	.0405	1.28

On the opposite page is a table by the same observer, which gives the properties of waves under varying conditions of the weather.

Measurements of Waves.—The usual measurements of waves include the elements length, height, and celerity.

When a ship is steaming directly across a system of regular waves, the length can be measured by aid of a line towed astern with a buoy (or other object that can be readily distinguished) at the end. The length of the line can be adjusted so as to bring the buoy on the crest of a wave at the time that the next crest is under the

PROPERTIES OF WAVES IN VARIOUS CONDITIONS OF WEATHER.

State of the Sea.	Velocity of Wind, V, Knots per H.	Period of Waves in Seconds,	Length in Feet.	Height in Feet, $h = 2r$.	Speed, C.		Ratio $\frac{h}{L}$.	Ratio $\frac{V}{C}$.
					Feet per Sec.	Knots per Hour.		
Smooth sea	11	5.7	203	5.2	35.4	21.0	.0258	0.51
Slight waves	11	6.5	256	7.9	39.0	23.1	.0308	0.50
Larger waves	17	8.7	394	13.4	45.2	26.8	.0342	0.67
Rough sea, clapotic	25	6.2	252	11.6	41.0	24.3	.0461	1.07
Heavy sea	40	7.6	347	16.6	54.8	32.4	.0476	1.43
Very heavy sea	58	8.6	485	25.4	56.4	33.4	.0524	1.66

stern of the ship. A log-line marked off by knots will be found convenient for the purpose, as the lengths can be read directly from the knots which are tied at intervals of 50.7 feet. Since waves are seldom quite regular, and lengths between successive crests may vary, there may be difficulty in adjusting the length of the line. In such case it may be convenient to note the position of a crest along the hull of the ship at the time the buoy is on the crest of the next wave. Large ships will often be longer than the waves measured, and then their length can be determined by noting the positions on the hull of two successive crests. If the ship is steaming at a small angle with the direction of the waves, the observed distance between crests is to be multiplied by the cosine.

Admiral Pâris* describes a device for tracing the forms of waves automatically, consisting of a long pole, or mast, ballasted to float erect and with a disk at the lower end which acts in a manner like a sea-anchor to prevent vertical oscillation of the pole. At the surface of the water is an annular buoy surrounding the pole and moving up and down with the surface of the water, which is connected with a registering device by a proper reducing motion, so that it actuates a pencil that moves over a band of paper which is driven by clockwork. The clockwork also registers seconds on the paper, thus giving means for determining the length, height, and time of the wave, as well as recording the contour of the wave profile.

If a ship is lying at rest across the crests of regular waves, the celerity can be determined by noting the time required to run the

* Description et usage du trace-vague. Revue Maritime et Coloniale, juin 1867.

length of the ship; if the ship has a speed of v feet per second against the waves, then in the time t that it takes the crest to run from the stem to the stern the ship will have moved vt feet; and if l is the length of the ship, the speed of the wave will be

$$C=\frac{l-vt}{t}=\frac{l}{t}-v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the course of the ship makes an angle α with the direction of motion of the waves, the celerity will be

$$C=\left(\frac{l}{t}-v\right)\cos\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the ship is running with the waves, the sign of v will be reversed.

The following method of estimating the heights of waves requires judgment and practice, but does not require instruments: the observer finds by trial a position at a proper height, such that he can just see the horizon over the crest of the nearest wave when the ship is erect in a hollow. The distance of his eye from the water is then the height of the wave from crest to hollow. The position of the observer depends on the height of the wave, and may be at a port, on the deck, on a bridge, or in the shrouds. A short pendulum may aid the observer in determining when the ship is erect; with or without such an aid, it will not be easy for the observer to determine that the ship is erect, that it is in the hollow, and that his eye is at the proper height. Care must be taken not to confound the crest of a distant wave with the true horizon.

Waves in Shallow Water.—The investigation of waves in shallow water may be made to depend on the equations

$$\frac{\phi}{c}=-x+\beta[e^{k(z+d)}+e^{-k(z+d)}]\sin kx, \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\psi}{c}=-z+\beta[e^{k(z+d)}-e^{-k(z+d)}]\cos kx, \quad . \quad . \quad . \quad . \quad (2)$$

provided that the height of the waves is small compared with the length; ϕ is the velocity potential, ψ is the stream function, β and

k, as before, are arbitrary constants, and d is the depth of the water. The origin of coordinates is taken at the level of the undisturbed water, which is also assumed to be the medial line of the wave profile, since the height of the wave is very small. As in the previous discussion of irrotational waves, the crests of the waves are assumed to have a velocity $-c$ feet per second, and the wave motion is assumed to be superposed on a uniform flow with a velocity c, so that the profile of the free water-surface is stationary.

The existence of a velocity potential infers irrotational motion.

To show that the stream function corresponds to the velocity potential we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial z}=-c\{1-k\beta[e^{k(z+d)}+e^{-k(z+d)}]\}\cos kx=-u, \quad . \quad (3)$$

$$\frac{\partial\phi}{\partial z}=-\frac{\partial\psi}{\partial x}=ck\beta[e^{k(z+d)}-e^{-k(z+d)}]\sin kx=-v, \quad . \quad . \quad . \quad (4)$$

as required by equations (28), page 250.

The condition for equilibrium may be expressed by equation (16), page 268:

$$\frac{g}{w}p=\text{const.}+\frac{\partial\phi}{\partial t}-gz-\tfrac{1}{2}(u^2+v^2); \quad . \quad . \quad . \quad . \quad (5)$$

and here for steady motion it is evident that ϕ is not a function of t, so that $\frac{\partial\phi}{\partial t}$ is zero. The values of u and v may be taken from equations (3) and (4), and for an approximation we may reject terms containing $k^2\beta^2$, and further in the exponents of e we may neglect z, which is small compared with h, thus reducing equation (7) to

$$\frac{g}{w}p=\text{const.}-gz-\tfrac{1}{2}c^2\{1-2k\beta(e^{kd}+e^{-kd})\}\cos kx. \quad . \quad . \quad (6)$$

At the free surface of the profile of the water is a stream-line subjected to the constant pressure of the atmosphere. For a stream-line ψ in equation (2) is a constant and for the free surface we may make $\psi=0$. Assuming as before that z is small, the

equation of the wave profile at the surface may be written

$$z=\beta(e^{kd}-e^{-kd}) \cos kx, \quad \ldots \quad (7)$$

which is the equation of a curve of sines with a crest at the origin and a hollow at

$$x_1=\tfrac{1}{2}L=\pi \div k,$$

so that the length of the wave from crest to crest is

$$L=\frac{2\pi}{k}; \therefore k=\frac{2\pi}{L}. \quad \ldots \quad (8)$$

Since the wave profile is a curve of sines with the medial line at the undisturbed water-surface, it may be considered that there is no discontinuity.

Now multiply the term in equation (6) which contains e^{kd}, by z and divide by its equivalent in equation (7), and also unite the constant term $\frac{1}{2}c^2$ with the general constant, thus getting

$$\frac{g}{w}p=\text{const.}+\left(kc^2\frac{e^{kd}+e^{-kd}}{e^{kd}-e^{-kd}}-g\right)z. \quad \ldots \quad (9)$$

The only condition that will make p constant is expressed by the equation

$$c^2=\frac{g}{k}\frac{e^{kd}-e^{-kd}}{e^{kd}+e^{-kd}}, \quad \ldots \quad (10)$$

which expresses the speed of the wave in quiet water in terms of the length of the wave and in the depth of the water.

If d is large compared with L, this equation reduces to that for the speed of waves in deep water, equation (16), page 255, that is, to

$$c^2=\frac{g}{k}. \quad \ldots \quad (11)$$

To illustrate the effect of depth on the speed of waves it may be noted that when the depth is equal to one-fourth of the length, the velocity is 0.96 of that in deep water; and that when the depth is half the length, the velocity is 0.9998 of that in deep water.

It is interesting to determine the greatest velocity of waves of the sinusoidal type in water of the depth d. For this purpose equation (10) may be written

$$c^2 = gd\frac{1}{kd}\frac{e^{kd} - e^{-kd}}{e^{kd} + e^{-kd}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and then expanding the exponentials and rejecting terms containing higher powers than the squares,

$$c^2 = gd\frac{1}{kd}\frac{1 + kd + \frac{1}{2}k^2d^2 - 1 + kd - \frac{1}{2}k^2d^2}{1 + kd + \frac{1}{2}k^2d^2 + 1 - kd + \frac{1}{2}k^2d^2}; \quad . \quad . \quad . \quad (13)$$

$$\therefore \ c^2 = gd\frac{2}{2 + k^2d^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which shows that an undulating wave in water having the depth d may approach but cannot reach the speed

$$c = \sqrt{gd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

It will appear later that the velocity given by equation (15) is proper for a certain isolated wave, called a solitary wave, when its height is small compared with the depth of the water; the conclusions that can be drawn from this coincidence will be found interesting in connection with the discussion of waves which accompany ships at high speeds.

To complete our discussion of waves in shallow water, we must investigate the conditions at the bottom. For this purpose let z in equation (2) be made equal to $-d$; this gives

$$\frac{\psi_0}{c} = d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which shows that a particle at the bottom moves along a straight line coincident with the bottom, as, of course, should be the case. It may be noted also that equation (4) shows that when z is equal to $-d$, the velocity v is always zero, which is another way of showing the same thing.

Superposition of Trochoidal Waves.—The following geometrical method, which is duc to Rankine, allows us to superpose trochoidal

waves which have the same length; the heights may be equal or unequal and the waves may run in the same or in contrary directions.

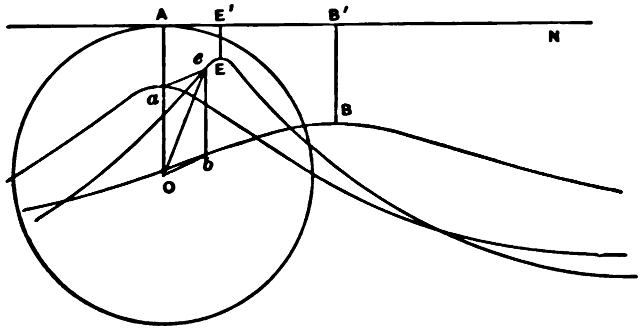


FIG. 136.

Take, first, waves running in the same direction and having the length L and the heights $h_1 = 2r_1$ and $h_2 = 2r_2$, and having the second crest a distance $\frac{1}{n}L$ from the first. In Fig. 136 take the circle with radius OA for the rolling circle for all three waves, the two component waves and the resultant wave after superposition; this, of course, is contrary to the theory of trochoidal waves, which (page 264) requires that a particle of water affected by a trochoidal wave motion shall be at the height $\pi r^2 \div L$ above its position when at rest; and as the latter location must be the same for all three waves, the centre of oscillation for the particle when affected by these several waves cannot be at the same height. The method is approximate only, and should properly be applied only to waves which have a small height compared with the length.

In Fig. 136 let O be the centre of the rolling circle for all three waves (the two component and the resultant waves), and let AN be the line along which it rolls. Take a point a at the crest of a wave which has the height $h_1 = 2r_1$; lay off the angle AOb, equal to $\frac{2\pi}{n}$, and take the point b at the distance $r_2 = \frac{1}{2}h_2$ from the centre O of the rolling circle; then b will be the describing point for the second wave, which will clearly have its crest at the distance AB', equal to $\frac{1}{n}L$ from the crest of the first wave. Draw the paral-

lelogram $Oaeb$; then the describing point of the resultant wave will be at e, and the radius of the orbit of that wave will be $r=Oe$. The height of the component wave is twice Oe and may be represented by

$$h=2r=2\left(r_1^2+r_2^2+2r_1r_2\cos\frac{2\pi}{n}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore h=\left(h_1^2+h_2^2+2h_1h_2\cos\frac{2\pi}{n}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the crests of the waves coincide, then $\frac{1}{n}=0$ and $h=h_1+h_2$; if the hollow of the second wave coincides with the crest of the first, then $\frac{1}{n}=\frac{1}{2}$, and $h=h_1-h_2$.

It is clear from inspection of Fig. 136 that the crest of the resultant wave will lie nearer the crest of the larger component wave; if the waves have equal heights, the crest will be midway between the crests of the component waves. To get the true location of the crest of the component wave we may proceed as follows: First find the angle AOe by the proportion

$$r:r_2::\sin AOb:\sin AOe,$$

whence

$$\sin AOe=\frac{r_2}{r}\frac{2\pi}{n}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and then find AE' by the equation

$$AE'=R\,.\,AOe=\frac{L}{2\pi}AOe. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the waves to be combined have contrary directions, they may be combined as in Fig. 137. Let c be the centre of orbit of a particle which under the influence of a wave running to the right would describe a circular orbit with the radius ca. Suppose that it is at the

same time affected by a wave motion which is moving toward the left and which would cause it to describe a circular orbit about c with the radius ae; lay off the angle zca' to the right and draw the radius $ca' = ca$; at a' draw the angle $z'a'e' = zca'$ toward the left, and draw the radius $a'e'$ equal to ae: then will e' be a point of the orbit of the particle. It is immediately evident that the orbit of the particle is an ellipse with the major axis vertical, and that all particles where crests correspond with crests (or hollows with hollows) will have such elliptical orbits with vertical major diameters. A particle a quarter of a wave length to the right of c (as in Fig. 138) will have an elliptical orbit with the major diameter horizontal, because then the crest of each component wave will have run a quarter of a wave length, one to the right and one to the left, and the crest of one wave will come opposite the hollow of the other wave, and the height of the resultant wave will be the difference of the heights of the component waves. A particle an eighth of a

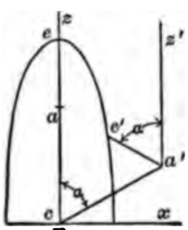


FIG. 137.

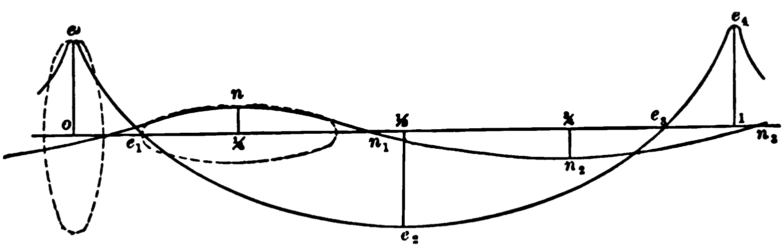


FIG. 138.

wave length from C will have an elliptical orbit with its major axis turned to an angle of 45° with the horizon, and in like manner the orbit of every particle will be an ellipse with the major axis at its appropriate angle. To get the contour of the surface of the water affected by such a component wave motion at any instant draw a curve through corresponding points in the elliptical orbits such as e, e_1, e_2, e_3, and e_4 for the condition when crests correspond to crests. It will, of course, take the wave a time

$$T = L \div c$$

to run a wave length, as is evident from equation (44), page 261; after a time $\frac{1}{4}T$ the crests will correspond to hollows and the contour will take the form nn_1n_2. If a succession of wave profiles be drawn, it will appear that the resultant wave will have a crest which runs in the direction of the higher component wave, and this crest will alternately grow and diminish; at certain places half a wave length apart the crests of the resultant wave will be high and sharp, and at intermediate places the crests will be low and obtuse.

If the heights of the component waves are equal, the ellipses of Figs. 137 and 138 become straight lines and there is no apparent progressive wave motion. At any instant the profile of the water-surface is like that of a trochoidal wave, but the profile, instead of appearing to run to the right or left, will grow from a horizontal surface, attain a maximum development, and then flatten out till the surface is again horizontal; immediately another wave profile will form with its crests where the hollows formerly were, will grow and flatten out, etc. If attention is concentrated on a certain crest, it will be seen to grow to its greatest height, die away, and be succeeded in the same place by a hollow, and the interval of time between the successive formations of crests at a given place will be the same as the time of one of the component waves. This action is most clearly seen where a wave is reflected from a vertical sea-wall, and is known as the clapotis. Waves which roll up on a sloping beach and run back again produce a motion similar to that of the clapotis, which is likely to be confusing and dangerous.

Group Velocity.—It has often been noticed that when an isolated group of waves is advancing over relatively deep water, the velocity of the group is less than that of the waves composing it. If attention is fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, while its former position in the group is occupied in succession by other waves which have sprung up in the rear.

The simplest investigation of this phenomenon is by the superposition of two systems of waves which have the same height, and nearly but not quite the same length and speed. Since the lengths and speeds of the component waves are unequal, the graphical method cannot be used, nor can the device of steady motion. But

if the heights are very small, there are some simple relations that are convenient for the present purpose.

Let the equation of equilibrium take the form of (16), page 268:

$$\frac{g}{w}p = \text{const.} + \frac{\partial\phi}{\partial t} - gz - \tfrac{1}{2}(u^2+v^2). \quad . \quad . \quad . \quad . \quad (1)$$

If the only velocity of a particle is that due to the wave motion, and if the orbit traversed by a particle is very small, then the square of that velocity represented by $\frac{1}{2}(u^2+v^2)$ may be neglected compared with the other terms in the equation. At the free water-surface the pressure is constant and equal to that of the atmosphere; consequently the only condition that is consistent with equation (1) is

$$z = \frac{1}{g}\frac{\partial\phi}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the origin of coordinates being taken at the undisturbed free water-surface. In equation (2) z is the ordinate of a particle of water at the free water-surface, and that equation is therefore an implicit equation of the profile of a wave.

A convenient equation for the velocity potential is

$$\phi = \frac{gr}{kc}\cos k(x-ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which t is the time and g is the acceleration due to gravity, while r, c, and k are constants; it will appear during the investigation that r is the half-height of the wave and that c is the speed of the wave.

The existence of a velocity potential may be taken to indicate an irrotational motion; or it may be noted that, as ϕ is not a function of z, both sides of equation (21), page 246, are equal to zero. Deducing the partial differential coefficient in equation (2) from the last equation, we have for the equation of the wave profile

$$z = r\sin k(x-ct). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The speed of the wave is c; as can be seen by fixing the attention on a particle for which z is zero, this particle will be found where

$$k(x-ct)=0,$$

that is, at the distance

$$x=ct,$$

from the origin, which shows that the rate of increase of x is c.

The profile of the wave at any instant is a curve of sines, as may be conveniently proved by making $t=0$, so that

$$z=r \sin kx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which differs from equation (46), page 277, only in that the wave crosses the axis at the origin instead of having a crest at that point.

All the properties of waves represented by equation (46) may be attributed to the form of waves under discussion, and in particular these waves may be considered to be equivalent to trochoidal waves which have a small height. It may also be considered that as the wave profile is a curve of sines with the undisturbed water-surface for the medial line, there is no discontinuity.

If two wave systems like that represented by equation (4) are superposed, the resultant profile may be represented by the equation

$$z=r \sin k(x-ct)+r \sin k'(x-c't) \quad . \quad . \quad . \quad . \quad (6)$$

$$=2r \sin \frac{k(x-ct)+k'(x-c't)}{2} \cos \frac{k(x-ct)-k'(x-c't)}{2}$$

$$=2r \cos \left(\frac{k-k'}{2}x-\frac{kc-k'c'}{2}t\right) \sin \left(\frac{k+k'}{2}x-\frac{kc+k'c'}{2}t\right). \quad (7)$$

The equation to the profile of the resultant wave surface after superposition can be obtained by making t zero in equation (7), giving

$$z=2r \cos \frac{k-k'}{2} x \sin \frac{k+k'}{2} x. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If k were exactly equal to k', this equation would reduce to equation (5), which represents a curve of sines; but since these constants are unequal, the function

$$\cos\frac{k-k'}{2}x$$

will vary slowly from unity, where $x=0$, to zero, where

$$\frac{k-k'}{2}x_1=\tfrac{1}{2}\pi, \quad \text{or} \quad x_1=\frac{\pi}{k-k'};$$

and its value will again be unity where

$$x_2=\frac{2\pi}{k-k'};$$

so that the distance from the centre of one group of waves to the centre of the next group is

$$2\pi\div(k-k'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

On the other hand, if x is made equal to zero in equation (7),

$$z=-2r\cos\frac{k'c'-kc}{2}t\sin\frac{kc+k'c'}{2}t, \quad . \quad . \quad . \quad . \quad (10)$$

which gives the ordinate at the origin at any time t. At the time $t=0$ the amplitude of the wave crest passing the origin is $2r$, and this amplitude will again occur when

$$\frac{kc-k'c'}{2}t=\pi,$$

that is, after the time interval

$$t=\frac{2\pi}{kc-k'c'}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so that the expression (11) represents the time between the passage of the centres of two successive wave groups past the origin.

The group velocity is obtained by dividing the distance between two successive groups by the time between the passages of two successive groups past the origin; that is, by dividing expression (9) by (11), which gives for the speed of a group

$$s=\frac{kc-k'c'}{k-k'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If k' approaches k, the group velocity approaches

$$s=\frac{d(kc)}{dk}=c+k\frac{dc}{dk}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Now the speed of a trochoidal wave is, by equation (16), page 255,

$$c=\sqrt{\frac{g}{k}},$$

from which

$$\frac{dc}{dk}=-\frac{1}{2}\frac{g^{\frac{1}{2}}}{k^{\frac{3}{2}}},$$

and consequently

$$s=\tfrac{1}{2}c; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

so that the group velocity is half the speed of the individual waves of the group. This conclusion applies to waves in deep water which have a small height compared with their length.

This discussion applies to successive groups of waves, the distance between successive groups being $2\pi\div(k-k')$; but as the discussion proceeds on the assumption that k' approaches k without limit, the distance between groups approaches infinity and the conclusion may therefore be applied to an isolated group of waves.

Capillary Waves.—The discussion of waves has proceeded on the assumption that water is a perfect fluid without cohesion, and that the only influence of the atmosphere is to produce a uniform pressure

on the free water-surface. An investigation of the effect of surface tension or capillarity indicates that this assumption is proper for waves that are long enough to affect ships, but in the investigation of the resistance of ships by aid of models it is important to determine the limiting conditions at which the effect of capillarity is appreciable. The discussion of capillary waves also throws some light on the generation of waves under the influence of the wind.

Lord Rayleigh * takes for the surface tension of pure water in contact with the atmosphere at 20° C. (68° F.) 74 dynes per linear centimeter. The surface of water which is affected by a simple wave motion has the crests parallel, and is therefore a cylindrical surface with the elements parallel to the crests. If the surface tension is represented by τ, the atmospheric pressure by p, and the pressure of the water at the surface by p', while R is the radius of curvature, then the resultant downward pressure at a crest is

$$p - p' = \frac{\tau}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

an equation like that for the tension of a thin hollow cylinder with internal fluid pressure.

In this discussion it is necessary to consider that both the air and the water are affected by a wave motion at the free surface. Strictly speaking, attention should be given to the elasticity of the air and the variation of its density with the varying pressure, but as the variations of pressure are small the density will be assumed to be constant, and the hydrodynamic equations for equilibrium and continuity will be applied to the air as well as the water.

Two equations for velocity potential are needed, one for the air and the other for the water; convenient forms are

$$\text{(air)} \qquad \phi' = C'e^{-kz} \cos kx \cos kct, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{(water)} \qquad \phi = Ce^{kz} \cos kx \cos kct. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* On tension of water-surfaces, etc., Phil. Mag., Nov. 1890.

The equation for equilibrium for the water may be written as on page 268:

$$\frac{g}{w}p = \text{const.} + \frac{\partial\phi}{\partial t} - gz - \frac{1}{2}(u^2+v^2).$$

If the height of the wave is very small, the orbit of a particle will be very small and its velocity in the orbit will be small so that the square of the velocity can be neglected in the above equation, giving

$$\frac{g}{w}p = \frac{\partial\phi}{\partial t} - gz + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A similar equation for the air is

$$\frac{g}{w'}p' = \frac{\partial\phi'}{\partial t} - gz + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If it be assumed that the pressure of the atmosphere is constant, the essential condition in order that equation (5) may hold is

$$z = \frac{1}{g}\frac{\partial\phi'}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and this is an implicit equation to the profile of the wave surface. Supplying a value of the differential coefficient from equation (2), the equation to the wave profile becomes

$$z = -\frac{ck}{g}C'e^{-kz}\cos kx \sin kct. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the wave has a very small height, this equation of the wave profile can be assumed to be of the form

$$z = r\cos kx \sin kct, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where r is the amplitude of the oscillation.

As with previous cases which give a curve of sines for the wave profile, it can be assumed that the conditions for continuity are fulfilled.

To find the relation of the amplitude r and the arbitrary constants in equations (2) and (3), we may resort to the following device: the

profile of a wave surface is a stream-line along which a particle of water or air may move, but which it cannot cross. Consequently the normal component of the velocity of a particle is the same as the normal velocity of the surface itself. But the slope of the wave surface is very small and the normal velocity makes a very small angle with the velocity, and therefore that velocity may be treated as though it were vertical. The vertical velocity of the wave's contour at any point may be obtained from equation (8), and is

$$\frac{dz}{dt} = kcr \cos kx \cos kct, \quad \ldots \ldots \quad (9)$$

and the vertical components of the velocity from equations (2) and (3) are

$$v = -\frac{\partial \phi'}{\partial z} = kC'e^{-kz} \cos kx \cos kct, \quad \ldots \quad (10)$$

$$v = -\frac{\partial \phi}{\partial z} = -kCe^{kz} \cos kx \cos kct. \quad \ldots \quad (11)$$

Since the height of the wave is small, the terms containing e may be replaced by unity, giving

$$v = -\frac{\partial \phi'}{\partial z} = kC' \cos kx \cos kct, \quad \ldots \ldots \quad (12)$$

$$v = -\frac{\partial \phi}{\partial z} = -kC \cos kx \cos kct. \quad \ldots \ldots \quad (13)$$

Comparing equations (12) and (13) with equation (9), it is apparent that

$$cr = C' = -C. \quad \ldots \ldots \ldots \quad (14)$$

Returning to equations (4) and (5), let the differential coefficients be obtained from equations (2) and (3), replacing the terms containing e by unity and taking account of the relation given by equation (14); this gives

$$\frac{g}{w'}p' = \frac{\partial\phi'}{\partial t} - gz + \text{const.} = -kc^2r\cos kx\sin kct - gz + \text{const.};$$

$$\frac{g}{w}p = \frac{\phi}{\partial t} - gz + \text{const.} = kc^2r\cos kx\sin kct - gz + \text{const.} \quad . \quad .$$

Replacing z by its value in equation (8),

$$\frac{g}{w'}p' = -(kc^2+g)r\cos kx\sin kct + \text{const.}; \quad . \quad . \quad .$$

$$\frac{g}{w}p = (kc^2-g)r\cos kx\sin kct + \text{const.} \quad . \quad . \quad . \quad .$$

The usual equation for the radius of curvature is

$$R = -\frac{\left\{1+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2z}{dx^2}} = -\frac{1}{\frac{d^2z}{dx^2}} \text{ (approximately)}; \quad .$$

the simplification being due to the fact that the slope $\frac{dz}{dx}$ of the surface is very small and that its square may be neglected. De ing the differential coefficient in this last equation from equatior

$$\frac{1}{R} = k^2r\cos kx\sin kct. \quad . \quad . \quad . \quad . \quad . \quad .$$

If now the values of p, p', and R are introduced into equation it becomes after reduction

$$\left(\frac{w}{g}+\frac{w'}{g}\right)kc^2 - \left(\frac{w}{g}-\frac{w'}{g}\right)g = \tau k^2, \quad . \quad . \quad . \quad .$$

and therefore

$$c^2 = \frac{w-w}{w+w'}\frac{g}{k} + \frac{\tau kg}{w+w'}. \quad . \quad . \quad . \quad . \quad . \quad .$$

In this transformation the constant terms disappear because it is assumed that the difference of pressure is entirely due to surface tension.

Since w' for air is small compared with w for water, equation (22) may be reduced approximately to

$$c^2 = \frac{g}{k} + \frac{\tau k g}{w}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

In the above equation c is the speed of the wave, as may be seen by inspection of equation (8), namely,

$$z = r \cos kx \sin kct.$$

If t in that equation is made $\frac{1}{2}\pi \div kc$, there will be a crest at the origin where $x = 0$, and another crest at the distance

$$x_1 = L = 2\pi \div k.$$

After the lapse of the time

$$T = 2\pi \div kc$$

another crest will come to the origin, and this is the time that it takes the crest of a wave to run a wave length L. Consequently the speed of the wave is

$$\frac{L}{T} = \frac{2\pi \div k}{2\pi \div kc} = c.$$

All this might, of course, be inferred directly from the discussion of irrotational waves on page 275, since that form of wave approaches the curve of sines when the height is very small.

The value of τ already quoted is 74 dynes to the linear centimeter; and to correspond with this quantity $g = 981$ centimeters per second, the density of water is 981 dynes per cubic centimeter, so that

$$\frac{w}{g} = 1 \quad \text{and} \quad \frac{w'}{g} = 0.001293.$$

If τ is made zero in equation (23),

$$c^2 = \frac{g}{k}, \quad \ldots \ldots \ldots \ldots \quad (24)$$

as for trochoidal waves and irrotational waves with small height, under the influence of gravity only.

If the first term in equation (23), which depends on gravity only, becomes small compared with the second term, which depends on surface tension, the speed of the wave becomes

$$c^2 = \frac{\tau k g}{w} = \frac{2\pi\tau g}{wL}, \quad \ldots \ldots \ldots \quad (25)$$

which shows that capillary waves increase in speed as the length diminishes. This is illustrated by the following table in centimeters and seconds:

Wave Length.	Speed.	Frequency.
0.50	30	61
0.10	68	680
0.05	96	1930

An interesting result of this investigation is the fact that there is a minimum speed for waves under the combined influence of surface tension and gravity. This may be found by equating to zero the first differential coefficient of c with regard to k obtained from equation (23). This makes

$$k_0^2 = \frac{w}{\tau} \quad \ldots \ldots \ldots \ldots \quad (26)$$

and

$$L_0 = \frac{2\pi}{k_0} = 2\pi\sqrt{\frac{\tau}{w}} \quad \ldots \ldots \ldots \quad (27)$$

In the C. G. S. system, which has been used for stating the surface tension,

$$L_0 = 2\pi\sqrt{74} = 1.73 \text{ centimeters,}$$

or 0.68 of an inch.

The corresponding speed is

$$c = \sqrt{2}\sqrt[4]{g\tau} = \sqrt{2}\sqrt[4]{981 \times 74} = 23.3$$

centimeters per second, or 0.76 feet per second, or 0.45 of a sea-mile per hour. This result agrees with an observation by Scott-Russell that a wind with a velocity of half a mile an hour can start a ripple on the surface of still water; this ripple, however, soon disappears if the breeze dies down. The same observer says that true waves appear when the wind has a velocity of two miles an hour, which is the speed of waves 2.2 feet long.

The most important application of the investigation of this discussion of capillary waves is to the determination of the proper length of models for experiment by Froude's method in a towing-tank. Equation (23) may be written

$$c^2 = \frac{g}{k}\left(1 + \frac{\tau k^2}{w}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

the second term in the parenthesis showing the effect of surface tension. In order that the effect of that term on the velocity may not be greater than one per cent. its absolute value must not be more than 0.02. Using the C. G. S. system as before,

$$\frac{\tau k^2}{w} = 0.02, \quad \text{or} \quad k = \sqrt{\frac{0.02 \times 981}{74}}$$

so that

$$L = 2\pi \div k = 2\pi\sqrt{\frac{74}{0.02 \times 981}} = 12.2$$

centimeters, or 4.9 inches. Dropping the term in equation (28), which depends on surface tension, the corresponding speed is

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{gL}{2\pi}} = \sqrt{\frac{981 \times 12.2}{2\pi}} = 43.7$$

centimeters per second, or 0.78 of a knot per hour. This is the slowest speed at which a model may be towed if the wave-making resistance is not to be affected by an error of more than one per cent. on account of the influence of surface tension. Suppose that models are to be tested at the speed corresponding to one fourth the full speed of the ship; then the model should not have a full speed of less than three knots per hour. If the length of the model is made 12 feet and the speed of the ship is made 21 knots per hour, then the ship may be $7^2=49$ times as long as the model, which will make the length of the ship 588 feet—not an unusual proportion.

Waves in Shoaling Water.—When waves run from deep water into shoaling water the following phenomena are observed: (1) The kinetic energy is concentrated in or transmitted to a shallower mass of water resulting in a more rapid revolution of the particles and increase in the height of the waves. (2) The length of the waves is diminished. (3) In consequence of the decreased length and increased height waves are more liable to break at the crest, and this tendency is increased by the fact that the wave is checked at the bottom more than at the surface. (4) When the waves break, the broken water is thrown forward and runs up onto the beach. (5) Return waves of broken water start from the beach and meeting incoming waves give rise to clapotic action.

A gentle swell in deep water may become troublesome or dangerous where it runs onto a shoal; if the depth is much reduced on the shoal, waves may break although the shoal may be entirely covered with water. In like manner a sunken ledge may cause waves to break. Very long waves have been known to break in a depth greater than 150 feet, and the influences of the coast may be felt to the distance of fifty miles or more.

The Solitary Wave.—Suppose that at one end of a canal there is a movable partition as shown in Fig. 139, behind which water may be stored above the level in the canal, as shown by the line EE'. If the partition be raised rapidly, the excess of water $DAFE$ above the level in the canal will form a wave of peculiar nature, as represented by *abcd*, which will travel forward along the canal. If the canal is closed at the father end by a vertical wall, the wave will be reflected and will return with the height but little diminished, provided the

canal is short, and the wave on its return can be caught behind the partition, in part at least.

There are other ways of producing a solitary wave. For example, a plate as wide as the canal may be placed against one end as at *DC*, and then moved quickly forward to a distance which will dis-

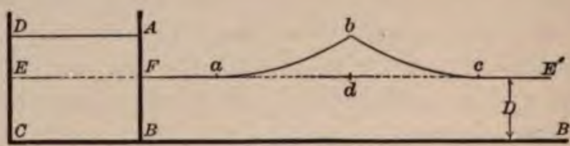


FIG. 139.

place a volume equal to *DAFE*, or a solid block equal in volume to *DAFE* may be thrust down into the canal and may displace the water required to form the wave.

The wave made in either of these ways is entirely above the surface of the undisturbed water in the canal, as shown by *abcd* in Fig. 139; it is called a positive solitary wave. If a hollow is formed at the end of the canal by drawing back a plate or by removing a block, then a hollow wave entirely below the surface of the undisturbed level in the canal will be formed and will travel forward as the positive wave does. Such a solitary wave is called a negative wave; its characteristics are like those of the positive wave, but it is less stable and is not so well known.

Laws of the Solitary Wave.—The following laws of the solitary wave are due to Scott-Russe l,* the discoverer of that form of wave:

(A) The profile of the positive solitary wave is entirely above the quiescent surface of the water in the canal; the profile of the negative wave is below the surface of the water.

(B) The surface of the wave is cylindrical, with elements perpendicular to the length of the canal.

(C) All molecules in a transverse vertical plane remain in a plane when acted upon by the wave; they move up and down in a plane, and the plane is transported forward by the positive wave

* Modern System of Naval Architecture, 1865.

a distance e, which is equal to the distance through which a plate placed against the end of the canal must be moved forward to generate the wave. A negative wave transfers molecules backward.

(D) The trajectory of a molecule is a semi-ellipse with its axis horizontal and equal to e. The vertical semi-axis is a maximum at the surface, and decreases to zero at the bottom, according to the law

$$f=\frac{F}{D}z_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which f is the semi-axis for a molecule at the height z_0 from the bottom when at rest, F is the semi-axis at the surface, and D is the depth in the canal.

(E) The volume of the intumescence produced by the wave is equal to the volume of the water which produces the wave, as is required by the condition of continuity. If V is the volume of the intumescence above the quiescent surface of the water, and W is the width of the canal, then

$$V=eDW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Further, if A is the area of the profile above the surface of the water at rest, then

$$AW=V=eDW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This law makes the form of the wave correspond to the requirement of continuity; the requirements for equilibrium are not met, as will be found if an attempt is made to apply the equations of equilibrium. Modern writers on hydrodynamics do not mention Scott-Russell's laws, probably on account of this deficiency. The defect in his laws affects the form of the profile as constructed by the method given in the next section. It probably does not affect the law for speed, which is the most important feature for the purposes of the naval architect, and fortunately the one that is most easily measured.

(F) The profile of the wave may be drawn as follows:

In Fig. 140 lay off a horizontal line $ON=L_0=2\pi D$, and on it draw a sinusoid $OdlfN$, making $lq=F$. From O lay off $Oh=e$ and draw

on it a semi-ellipse Oah with the semi-minor axis equal to F. From any point as a on the semi-ellipse draw a horizontal line acg intersecting the vertical hi, through h, at c, and the sinusoid $OdlfN$ at d. From d lay off $de = ac$; then will e be a point on the profile of the wave. On the same line ag there is another point of the profile at g found by making $fg = bc$; so that the line eg is less than the line df by the length of the line ab. As a consequence the area of the figure hmN is less than the area of the figure OlN, by the area of the semi-

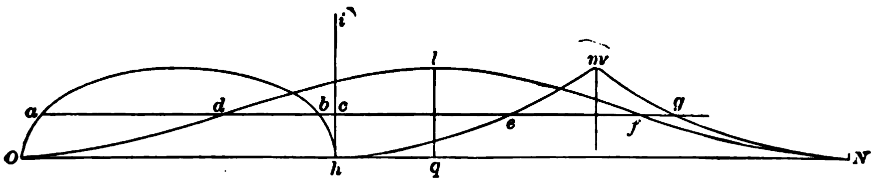


FIG. 140.

ellipse Oah. This property gives the readiest way of finding the area of the section of the intumescence.

From the symmetry and method of construction of the sinusoid its area is known immediately to be πFD. The area of the semi-ellipse is $\frac{1}{4}\pi eF$, consequently the area of the intumescence is

$$A = \pi F\left(D - \frac{e}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Comparing equation (4) with equation (3), it is apparent that

$$F = \frac{eD}{\pi\left(D - \frac{e}{4}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, it is apparent from the construction of Fig. 140 that the length of the wave is

$$L = L_0 - e = 2\pi D - e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the height of the wave is very small compared with its length, so that e is small compared with H, then the equations for the area, the height, and the length approach

$$A=\pi FD \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$F=\frac{e}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$L=2\pi D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The profile of the wave in such case is sensibly the sinusoid itself instead of the derived curve.

(G) The height of the wave cannot exceed the depth of the canal; at that limit the wave will break.

If F be made equal to D in equation (5),

$$e_m=\frac{\pi D}{1+\frac{\pi}{4}}=\tfrac{4}{7}\pi D \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From equation (6),

$$L_m=2\pi D-e_m=2\pi D-\tfrac{4}{7}\pi D.$$

$$\therefore\ L_m=\tfrac{10}{7}\pi D \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Also

$$A_m=e_mD=\tfrac{4}{7}\pi D^2 \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(H) The speed of the wave was found by experiment to be:

For the positive wave

$$c=\sqrt{g(D+F)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For the negative wave

$$c=\sqrt{g(D-F)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the height of the wave is small compared with the depth in the canal, then both equations reduce to

$$c=\sqrt{gD}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If the height of the wave becomes its maximum (equal to the depth in the canal), the positive wave has the speed

$$c=\sqrt{2gD}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

but the speed of the negative wave is zero.

(I) If the volume of water acted on to form a solitary wave exceeds the volume $A_m W$ of the wave having a height equal to the depth in the canal, then the maximum wave is formed and proceeds with its proper speed, and the remainder of the water used forms one or more smaller *residuary* waves, each having its own proper length and velocity. It is possible that an attempt to form a solitary wave may result in the formation of a positive wave followed by a negative wave, if the operation is performed too rapidly. In that case the positive wave will have a height corresponding to the speed impressed, and the negative will have its own proper speed, which will be less than that of the positive wave.

(K) The friction of the water against the sides and bottom of the canal, and also internal friction, will gradually use up the energy stored in a solitary wave, and will dissipate it. The height of the wave gradually diminishes, and with it the speed as a natural consequence. But as the speed is proportional to the square root of the depth of the canal plus the height of the wave, the speed diminishes less rapidly than the height.

Scott-Russell's attention was first directed to the solitary wave which accompanied a canal-boat drawn by horses. He noticed that the wave became detached and proceeded by itself along the canal when the boat stopped. At one time he followed such a wave a considerable distance on horseback. At another time, by sending boats all in one direction each carrying its solitary wave with it, he raised the level perceptibly at the farther end of the canal.

Waves Accompanying Ships.—A large vessel proceeding along the channel of a river will raise a solitary wave which will precede the vessel. Usually the speed of the ship is low and the wave is interesting mainly because it washes the banks.

The resistance of a vessel at high speed is due in large part to the energy which it must expend in maintaining the train of

waves which it carries with it. This matter will be discussed in connection with the resistance of ships in a future chapter. It is of interest now to note the changes that occur in the waves accompanying such a vessel when the depth of the water changes. A torpedo-boat running at its maximum speed can maintain that speed in practically all depths of water by exerting about the same horse-power. If the boat is in deep water, the length of the waves accompanying it (from crest to crest) will be given by the equation

$$L_1 = \frac{2\pi c^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is derived from equations (43) and (44), page 260; this equation is sufficient until the depth is less than half L; then it will be proper to use the equation

$$L_2 = \frac{2\pi d}{\left(2\frac{gd}{c^2} - 2\right)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

derived from equations (10) and (14), pages 282 and 283. The length of the wave clearly increases as the water becomes shallower, and when the depth becomes as little as

$$D_1 = \frac{c^2}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where c is the speed of the boat, the train is reduced to one wave, i.e., to the solitary wave which has only a small height compared with the depth of water, as is seen by comparing with equation (15), page 303. When the depth decreases to

$$D_2 = \frac{c^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(compare with equation (16), page 304), the solitary wave has its maximum speed for the depth of water, and if the boat runs at full speed into shallower water, it is not accompanied by any waves. This succession of events, which is stated broadly without attention

to details and without reservations which such attention to details would suggest, can be attributed only to boats which, like torpedo-boats, can be driven at very high speeds. Large ships never are driven at correspondingly high speeds, and cannot maintain full speed in shallow water, where they meet with much greater resistance than in deep water.

CHAPTER IX.

ROLLING OF SHIPS.

THE oscillations that can be impressed on a ship are distinguished as (1) rolling, (2) pitching and (3) heaving. Simple rolling consists of an oscillation about a longitudinal horizontal axis; simple pitching is an oscillation about a transverse horizontal axis; heaving is a vertical oscillation of the centre of gravity of the ship. When one of these oscillations is communicated to a ship, one or both of the other two are liable to occur also. Thus, when a ship is set to rolling by men running across the deck, it pitches and heaves also.

In a resisting medium like water the rolling is resisted by the friction of the water on the skin of the ship, by direct resistance of the form of the ship (especially by the keel and bilge-keels), and by the waves set up. These several resistances can usually be combined so as to give a resultant couple and a resultant force, which latter causes a translation of the centre of gravity of the ship. If the ship rolls without pitching or heaving, the resultant force is horizontal and transverse. It changes its intensity and direction as the ship rolls, and gives rise to a horizontal transverse oscillation of the centre of gravity of the ship. Pitching and heaving are resisted in a like manner; pitching gives rise to a longitudinal oscillation of the centre of gravity of the ship.

Each of the three oscillations (rolling, pitching, and heaving) has its own natural time, which time is affected by the extent of the oscillation; and usually the times do not agree and are not commensurable. It is apparent that the actual motions of a ship, especially when affected by the action of waves in a seaway, are exceedingly complicated. It is convenient to investigate first the oscillation of a ship which has such form and conditions that the several oscillations (rolling, pitching, and heaving) are independent, and to further

simplify the problem by assuming that the oscillations take place in a quiet unresisting medium; afterward the effects of resistance and of waves, and the interdependence of the several kinds of oscillation, will receive attention.

Rolling in an Unresisting Medium.—The rolling of a ship is more extensive and more important than pitching or heaving, and consequently receives more attention. To simplify the problem, it is customary to choose such a form for the ship that rolling will not induce pitching or heaving, and to assume that the rolling takes place in an unresisting medium.

If a ship is symmetrical fore and aft as well as transversely, then when the ship rolls the centre of buoyancy will remain in a vertical transverse plane through the centre of gravity of the ship, and rolling will not give rise to pitching.

To avoid heaving the form of the ship should be such that the centre of gravity will remain at a fixed height above the surface of the water. This will be the case when the centre of gravity is at the centre of curvature of the transverse curve of water-lines, provided that this curve is the arc of a circle.

The two conditions just assumed lead to rolling about an axis through the centre of gravity, for in an unresisting medium there is no resisting force to cause a horizontal translation of the centre of gravity.

Ships are not usually symmetrical fore and aft, and the centre of gravity is seldom at the centre of the curve of water-lines; but the deviation of ships from these conditions is not sufficient to give rise to any inconvenience in applying the results of the simplified investigations to ships of common form.

General Equation for Rolling.—When a ship is inclined to an angle θ the righting moment is

$$D(h-a)\sin\theta.$$

If the ship is rolling in an unresisting medium, this moment is applied to produce acceleration (or retardation) of the rolling which, in the simplified case, takes place about an axis through the centre of gravity.

The angular velocity of a ship may be represented by $\frac{d\theta}{dt}$, and the

angular acceleration by $\frac{d^2\theta}{dt^2}$. The linear acceleration at a given point at the distance ρ from the centre of gravity is

$$\rho\frac{d^2\theta}{dt^2}.$$

The force required to impart this linear acceleration to a mass dm at the given point is

$$\rho\frac{d^2\theta}{dt^2}dm,$$

and the moment of that force about the axis through the centre of gravity is

$$\rho^2\frac{d^2\theta}{dt^2}dm.$$

The moment required to impart the angular acceleration $\frac{d^2\theta}{dt^2}$ to the entire mass of the ship is

$$\frac{d^2\theta}{dt^2}\int\rho^2dm=\frac{I}{g}\frac{d^2\theta}{dt^2}, \qquad (1)$$

in which I represents the moment of inertia of the weight of the ship about the axis through the centre of gravity. If ρ_0 is the radius of gyration of the ship about the axis through the centre of gravity in feet, and if D is the displacement in tons, then

$$I=\rho_0{}^2D.$$

Equating the righting moment to the moment required to produce the angular acceleration (with contrary signs since they act in opposite directions),

$$D(h-a)\sin\theta=-\rho_0{}^2\frac{D}{g}\frac{d^2\theta}{dt^2},$$

whence

$$\frac{d^2\theta}{dt^2}=-\frac{g(h-a)\sin\theta}{\rho_0{}^2}. \qquad (2)$$

If this equation is integrated twice, it will give an expression for t, the time of rolling, in terms of the properties of the ship and the maximum angle of roll. But in general $h-a$, the arm of the righting couple, is given by a graphical curve of righting moments and cannot be expressed as a simple function of the angle of inclination. It will be shown later how the time of rolling for a given ship may be determined by a graphical process.

Isochronus Rolling.—In general the form of the metacentric curve, as MM_1 in Fig. 141, depends on the form of the ship, and can be determined by a proper calculation and construction from the lines of the ship. The form of the metacentric curve varies widely with the form of the ship, and by choosing a proper form for the ship any desired form of metacentric curve may be obtained. The admission that the metacentric curve can be chosen at pleasure leads to a very simple discussion of rolling, after which it is easy to obtain a sufficient idea of the influence that any form of metacentric curve will have on the rolling of the ship to which it belongs.

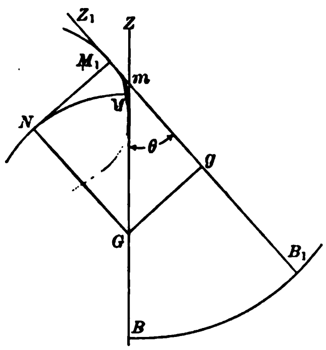


FIG. 141.

In Fig. 141 let the metacentric curve MM_1 be the involute of a circle which has its centre at G, the centre of gravity of the ship. In the same figure BB_1 is the curve of buoyancy, which is the involute of MM_1. It is not certain that the assumed construction for the metacentric will allow the curve of water-lines to be the arc of a circle; the error from this source will probably be small, and will be neglected.

In Fig. 141 the arm of the righting moment at the angle θ is Gg, drawn perpendicular to B_1Z_1, the new vertical. Drawing GN parallel and NM_1 perpendicular to the new vertical B_1Z_1, gives at once the point of contact of that vertical with the metacentric curve, since that curve is assumed to be the involute of the circle drawn through M with G as a centre. But

$$Gg = NM_1 = \text{arc } NM = NG\theta = (r_0 - a)\theta,$$

where $r_0 - a$ is the metacentric height GM. Consequently the righting moment of any angle is

$$D(r_0 - a)\theta.$$

Equating this righting moment to the moment required to produce the angular acceleration as expressed by equation (1),

$$D(r_0 - a)\theta = -\rho_0^2 \frac{D}{g} \frac{d^2\theta}{dt^2};$$

whence

$$\frac{d^2\theta}{dt^2} = -\frac{g(r_0 - a)}{\rho_0^2}\theta = -b^2\theta, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the constant b depends on the acceleration due to gravity, and on the properties of the ship $r_0 - a$ and ρ_0. This equation is readily integrated as follows:

$$\int 2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}dt = -b^2\int 2\theta\, d\theta.$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = -b^2\theta^2 + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At the end of a roll the ship comes to rest at the maximum inclination θ_m, and at that instant the angular velocity is zero, so that

$$C = b^2\theta_m^2;$$

and consequently

$$\frac{d\theta}{dt} = b\sqrt{\theta_m^2 - \theta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The time of a single roll is considered to be the interval between the maximum inclination on one side, and the maximum inclination on the other side; so that the time of rolling from the erect position (when $\theta = 0$) to the maximum inclination (when $\theta = \theta_m$) can be determined by integrating between the limits 0 and θ_m. Thus

$$\int_0^{\frac{1}{2}t} dt = \frac{1}{b}\int_0^{\theta_m} \frac{d\theta}{\sqrt{\theta_m^2 - \theta^2}}.$$

$$\therefore \frac{1}{2}t = \frac{1}{b}\left[\sin^{-1}\frac{\theta}{\theta_m}\right]_0^{\theta_m} = \frac{1}{b}\frac{\pi}{2}.$$

$$\therefore t = \frac{\pi}{b} = \frac{\pi\rho_0}{\sqrt{g(r_0 - a)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Another expression is commonly used in which ρ_0 is replaced by its value from the equation

$$I=\rho_0^2D,$$

which gives

$$t=\pi\sqrt{\frac{I}{gD(r_0-a)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which D is the displacement in tons and I is the moment of inertia of the weight of the ship in terms of tons and feet. These equations may be compared with the equation for oscillation of a simple pendulum:

$$t=\pi\sqrt{\frac{l}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Moseley's Graphical Method.—Starting from isochronous rolling with an involute for the metacentric curve, the time of rolling of a ship with any form of metacentric curve can be determined by aid of a graphical method due to Moseley. It is not necessary, however, to draw the metacentric curve in order to apply the method; it is sufficient to have the curve of statical stability.

In Fig. 142 let OM represent the curve of righting moments, or the curve of statical stability for the given ship; and let OW be the curve of dynamical stability corresponding. At a given angle of inclination θ, the righting moment is θm, and may be represented by m. Equating the righting moment to the moment required to produce the angular acceleration of rolling,

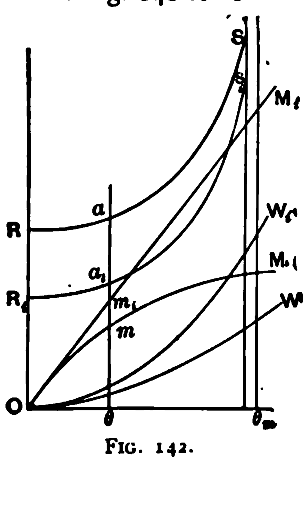


FIG. 142.

$$m=-\rho_0\frac{D}{g}\frac{d^2\theta}{dt^2}.$$

$$\therefore\ \frac{d^2\theta}{dt^2}=-\frac{g}{\rho_0 D}m=-\frac{g}{\rho_0^2 D}\frac{dw}{d\theta}. \quad (1)$$

in which w represents the ordinate Ow of the curve of dynamical stability which curve is the integral curve of the curve of statical stability. Integrating,

$$\int 2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}dt = -\frac{2g}{\rho_0^2 D}\int dw\,.$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = -b_1^2 w + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $b_1^2 = \frac{2g}{\rho_0^2 D}$, and C is the constant of integration which can be determined from the consideration that the angular velocity is zero when the inclination is a maximum, that is for θ_m. Consequently

$$C = b_1^2 w_m,$$

where w_m is the ordinate of the curve of dynamical stability at θ_m. Consequently

$$\frac{d\theta}{dt} = b_1\sqrt{w_m - w} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\int_0^{\frac{1}{2}t} dt = \int_0^{\theta_m} \frac{d\theta}{b_1\sqrt{w_m - w}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In Fig. 142, plot the curve RS with ordinates calculated by the equation

$$\frac{1}{b_1\sqrt{w_m - w}} = \frac{1}{\sqrt{\frac{2g}{\rho_0^2 D}}\sqrt{w_m - w}} = a. \quad . \quad . \quad . \quad . \quad (5)$$

Then

$$\tfrac{1}{2}t = \int_0^{\theta_m} a\,d\theta = A,$$

$$t = 2A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A is the area under the curve RS measured to the ordinate at θ_m. Unfortunately a_m, the ordinate of the curve RS at θ_m, is infinity, so that the area of the curve cannot be measured directly. But

the difficulty can be overcome by the following artifice: Suppose that we have a ship which has the same displacement, radius of gyration, centre of gravity, and metacentre as the actual ship, but which has for its metacentric curve the involute of a circle drawn from the centre of gravity through the metacentre, as shown in Fig. 141. The rolling of such a ship in an unresisting medium would be isochronous, and could be calculated by equation (6), page 311. The righting moment for such a ship is

$$D(r_0 - a)\theta,$$

and its curve of statical stability is a straight line like OM_i in Fig. 142. Let OW_i be the curve of dynamical stability corresponding, and let the curve R_iS_i be drawn with ordinates calculated by the equation

$$\frac{1}{\sqrt{\frac{2g}{\rho_0{}^2 D}}\sqrt{W_{im} - W_i}} = a_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The time of half an oscillation of the supposititious ship will be represented by the area under the curve R_1S_1 measured to the ordinate θ_m. On the other hand, the time of isochronous rolling can be calculated by equation (6), page 311, so that

$$t_i = 2A_i = \frac{\pi\rho_0}{\sqrt{g(r_0 - a)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which give a method of determining A_i numerically.

Now both the curve RS and the curve R_iS_i have the ordinate at θ_m for an asymptote, and if we draw an ordinate near θ_m, it will cut off nearly the same areas from the figures bounded by RS and R_iS_i. We may, therefore, consider that the area under RS exceeds the area under R_iS_i by the area RSS_iR_i, which can be measured by aid of a planimeter or otherwise. We shall then have for the approximate value of t,

$$t = t_i + 2RSS_iR_i = t_i + 2(A - A_i). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Influence of Form on Rolling.—All ships will have approximate isochronous rolling for small inclinations; in fact the deviation for ordinary forms and for moderate inclinations are insignificant, and

for practical purposes it is sufficient to calculate the time of rolling by equation (6), p. ooo, for any ship.

The graphical method just described allows us to determine from the metacentric curve of a given ship, or from the curve of statical stability, what the influence of form will be on unresisted rolling. Suppose that the metacentric curve for a given ship rises above the involute of a circle drawn from the centre of gravity through the metacentre, as shown by Fig. 143. Then for a given angle of inclination the righting moment is evidently greater than it would be for the supposititious ship with the involute for the metacentric curve. Then in a figure like 142 the curve OM will rise above the line OM_i, and consequently RS will lie below R_iS_i, so that equation (9) will take the form

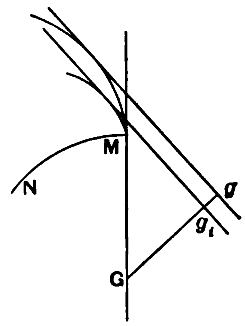


FIG. 143.

$$t = t_i - 2(A_i - A);$$

that is, the actual ship will have a shorter time of rolling than the supposititious ship. The deeper the ship rolls the quicker it will roll. But a comparison of the equations

$$D(h-a)\sin\theta$$

and

$$D(r_0 - a)\theta$$

shows that the line OM_i and the curve OM (in Fig. 142) very nearly coincide for small angles, because h is nearly equal to r_0 and sin θ is nearly equal to θ. More exactly, the line OM_i is the tangent to the curve OM at the origin. The conclusion is that when the curve of statical stability for a ship rises above the tangent to that curve at the origin, as represented by OM, Fig. 144, the ship will roll more quickly as it rolls deeper; on the contrary, the ship will roll slower as it rolls more deeply if its curve of statical stability lies below the tangent at the origin as shown by OM', Fig. 144.

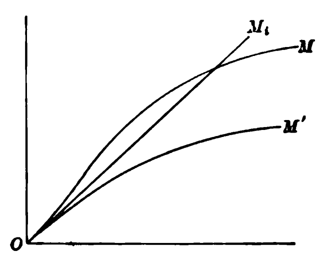


FIG. 144.

Wall-sided ships, and ships with a flare at the water-line come in the first class and roll quicker as they roll deeper. Certain ships in the French navy have a tumble-home above the water-line; they have a metacentric curve similar to that represented by Fig. 145, and they roll more slowly as they roll more deeply. This form is given to the ships in part to ensure certain nautical qualities, like slow rolling, and in part for convenience in arranging the battery.

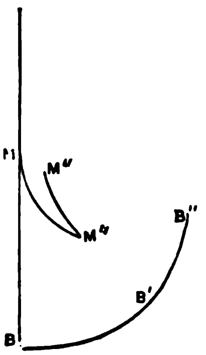


FIG. 145.

All the conclusions in this article refer only to rolling in an unresisting medium and to moderate angles (20° or 30°).

Pitching.—All ships are symmetrical transversely, and consequently simple pitching does not tend to move the centre of gravity of the ship out of a longitudinal plane; therefore, pitching does not give rise to rolling. Pitching may, however, give rise to heaving.

Since ships are not symmetrical fore and aft, an exact discussion of pitching would be more complex than the discussion of rolling. But the pitching of a ship is commonly limited to a small angle (10° or 12°), so that the expression for the righting moment may be written

$$D(H-a)\theta,$$

and the integration of the equation of righting moment and the moment required to produce angular acceleration,

$$\frac{d^2\theta}{dt^2},$$

gives

$$T=\frac{\pi\rho_0}{\sqrt{g(R_0-a)}};$$

in which ρ_0 is the radius of gyration of the ship about a transverse axis through the centre of gravity, and R_0-a is the longitudinal metacentric height.

Heaving.—If a ship is by any means forced deeper into the water than to the normal water-line, the added displacement produces

a vertical force which will cause the ship to rise when free to do so. Part of the work of the force may be absorbed by friction of water on the skin of the ship, or by any other resistance; the remainder will impart an acceleration to the ship. When the ship comes to the normal water-line the energy stored in it, due to the action of the vertical force just mentioned, will cause it to rise above the water-line. The ship will then rise and fall in the water till the energy imparted is absorbed by friction, by waves, etc. Rolling of a ship of common form is likely to be accompanied by both pitching and heaving; pitching will usually be accompanied by heaving, but not by rolling.

To simplify the discussion of heaving let it be assumed that the floating body has vertical sides and ends, so that the water-lines have a constant area of A square feet. If it be immersed x feet beyond the normal water-line, the vertical upward force due to the excess of buoyancy will be

$$\frac{xA}{35} \text{ tons.}$$

This force acting on the mass of the ship will produce an acceleration $\frac{d^2x}{dt^2}$, so that

$$\frac{xA}{35} = -\frac{D}{g}\frac{d^2x}{dt^2}. \quad \ldots \ldots \quad (1)$$

Rearranging and integrating,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}dt = -2\frac{Ag}{35D}xdx, \quad \ldots \ldots \quad (2)$$

$$\left(\frac{dx}{dt}\right)^2 = -b_2^2x^2 + C, \quad \ldots \ldots \quad (3)$$

where $b_2^2 = \frac{Ag}{35D}$. The velocity is zero at the maximum immersion x_m, consequently

$$C = b_2^2 x_m^2. \quad \ldots \ldots \quad (4)$$

Introducing the value and integrating,

$$\int_0^{\frac{1}{2}\tau} dt = \frac{1}{b_2}\int_0^{x_m} \frac{dx}{\sqrt{x_m^2 - x^2}}$$

gives the time for half a single oscillation, and

$$\tfrac{1}{2}\tau=\frac{1}{b_2}\left[\sin^{-1}\frac{x}{x_m}\right]_0^{x_m}=\frac{\pi}{2b_2}.$$

$$\therefore\quad \tau=\pi\sqrt{\frac{35D}{gA}}. \qquad (5)$$

Example.—A certain ship has a displacement of 3600 tons, a water-line area of 8900 square feet, a radius of gyration about a longitudinal axis of 19.7 feet, and about a transverse axis of 106.6 feet. The transverse metacentric height is 1.8 feet and the longitudinal metacentric height is 270 feet.

The time of rolling in an unresisting medium is

$$t=\frac{\pi\rho_0}{\sqrt{g(r_0-a)}}=\frac{\pi\times 19.7}{\sqrt{32.2\times 1.8}}=8.1 \text{ secs.}$$

The time of pitching is

$$T=\frac{\pi\rho_0}{\sqrt{g(R_0-a)}}=\frac{\pi\times 106.6}{\sqrt{32.2\times 270}}=3.6 \text{ secs.}$$

The time of heaving is

$$\tau=\pi\sqrt{\frac{35D}{gA}}=\pi\sqrt{\frac{35\times 3600}{32.2\times 8900}}=0.44 \text{ sec.}$$

It will be noticed that the pitching is about twice as fast as the rolling, and that the heaving is about eight times as fast as the pitching.

Each of the three oscillations is somewhat slower in a resisting medium like water.

Comparison of Rolling, Pitching, and Heaving.—The time of rolling of a ship is largely under the control of the designer, who may vary r_0-a by changing the height of the centre of gravity of the ship. The time of pitching can be only partially controlled, since any allowable change of the position of the centre of gravity will have very little effect on the value of R_0-a. In general it is desirable to keep weights away from the ends of ships to avoid slow and deep

pitching. There is a linear dimension in the numerator and the square root of a linear dimension in the denominator of the equation

$$T = \frac{\pi \rho_0}{\sqrt{g(R-a)}};$$

therefore the time of pitching increases with the size of ships of similar form, that is, larger ships pitch more slowly.

Again, the displacement is proportional to the cube, and the water-line area is proportional to the square of a linear dimension for similar ships; consequently larger ships heave more slowly.

Rolling Among Waves.—An approximate theory of unresisted rolling among waves due to Mr. William Froude* is based on the following assumptions:

1. The profile of the waves is assumed to be a curve of sines (or cosines).

2. The height of the waves is assumed to be small compared with the length.

3. The rolling of the ship in a calm, unresisting medium is assumed to be simple and isochronous.

4. The size of the ship is assumed to be small compared with the height of the waves.

5. It is assumed that the form of the wave is not influenced by the presence of the ship; this assumption may almost be considered to be a consequence of the preceding assumption.

As the ship moves up and down with the wave it will have the corresponding acceleration; but as the height of the wave is small, this acceleration will be small compared with the acceleration due to gravity, and will be neglected. The volume of water displaced by the ship will be assumed to be constant, and, further, the righting moment at any instant will be assumed to be the same as for the same inclination in still water except that now the inclination is to be measured from the normal to the wave profile instead of the absolute vertical.

A convenient form for the equation to the wave profile is

$$z = -r \cos (kx + kct), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Proceedings Int. Naval Arch., Vol. XIV.

from which we have

$$\phi = \frac{dz}{dx} = kr \sin (kx + kct) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

for the angle which the tangent makes with the axis of x at any instant; this is also the angle which the normal makes with the vertical. It is convenient to place the ship at the origin, so that equation (2) becomes

$$\phi = kr \sin kct. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The maximum angle between the normal and the vertical is

$$\Phi = kr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

so that equation (3) may be written

$$\phi = \Phi \sin kct. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

General Differential Equation.—In Fig. 146 let GZ_0 be the vertical through the centre of gravity G of a ship; let GZ be the normal to the wave profile, while GZ' represents the line of the ship's masts. If B is the centre of buoyancy of the ship which has a displacement of D tons, then the righting moment is, approximately,

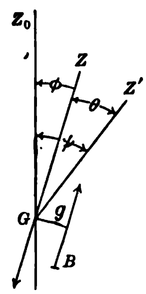


FIG. 146.

$$D \cdot Gg = D(r_0 - a)\,\theta. \quad . \quad . \quad . \quad . \qquad (6)$$

The angular acceleration per second from the vertical GZ_0 is

$$\frac{d^2\psi}{dt^2};$$

and following the method on page 309, the moment required to impress this angular acceleration on the ship is

$$\rho_0{}^2 \frac{D}{g} \frac{d^2\psi}{dt^2},$$

where ρ_0 is the radius of gyration of the ship and g is the acceleration due to gravity. Equating the righting moment to the moment required to produce the angular acceleration with the contrary sign,

$$\frac{d^2\psi}{dt^2} = -\frac{g(r_0 - a)}{\rho_0{}^2}\theta = -b^2\theta. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where b represents the coefficient of θ and has the same significance as in the discussion of rolling in an unresisting medium on page 311. But from Fig. 146 and equation (5) we have

$$\theta = \psi - \phi = \psi - \Phi \sin kct, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

so that equation (7) may be written

$$\frac{d^2\psi}{dt^2} + b^2\psi = b^2\Phi \sin kct. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This is the general differential equation for unresisted rolling among waves, with the approximations previously given.

In the application of this equation to the cases that may be advantageously discussed it is convenient to express the time of rolling of the ship in quiet unresisting medium and the time of the waves by the equations

$$2t = \frac{2\pi}{b} \quad \text{and} \quad T = \frac{2\pi}{kc}. \quad . \quad . \quad (10) \text{ and } (11)$$

Equation (10), which comes directly from equation (6), page 311, gives the time of a double roll from one side to the other and back again; and equation (11), which can be deduced from equation (31), page 258, gives the time from the passage of one crest to the passage of the next crest. Of the terms appearing in these equations c is the speed of the wave in feet per second, and $k = 2\pi \div L$, where L is the length of the wave in feet; while

$$b = \frac{\rho_0}{\sqrt{g(r - a)}},$$

in which ρ_0 is the radius of gyration of the ship, $r_0 - a$ is the metacentric height, and g is the acceleration due to gravity. If $2t$ is small compared with T, we have the case of quick rolling; if it is large, we have the case of slow rolling; and if $2t$ is equal to tT, the rolling is said to be synchronous.

Quick Rolling.—If the time of rolling of the ship is very small compared with the time of the waves, then the waves will not induce any proper rolling of the ship. which will follow the free surface of

the water with its masts coincident with the normal to the contour. A raft or a very shallow boat, like a harbor-defence monitor, will behave in this manner, and, though always wet at sea, will never have heavy masses of water on the deck.

The mathematical treatment of this case is obtained by solving equation (8) for ψ, obtaining

$$\psi=\theta+\Phi \sin kct, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

from which

$$\frac{d^2\psi}{dt^2}=\frac{d^2\theta}{dt^2}-k^2c^2\Phi \sin kct. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Substituting from equations (12) and (13) in the general equation (9),

$$\frac{d^2\theta}{dt^2}-k^2c^2\Phi \sin kct+b^2\theta+b^2\Phi \sin kct=b^2\Phi \sin kct;$$

and omitting terms which contain kc, since their value must be small when T is large,

$$\frac{d^2\theta}{dt^2}+b^2\theta=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which is the equation (3), page 311, for isochronous rolling in an unresisting quiet medium, except that now θ is the angle which the ship's masts make with the normal to the wave contour.

Slow Rolling.—If the time of rolling of a ship is large compared with that of the waves among which it may be placed, it will be steady and roll very little, if at all. A mathematical treatment may be produced which is the converse of the preceding; that is, b is now considered to be small compared with kc, which leads to the conclusion given; but such a treatment is unsatisfactory because only large ships may safely be given such properties as lead to slow rolling, and among short, quick waves they do not conform at all to the conditions laid down at the beginning of this chapter.

Synchronous Rolling.—The most important mathematical treatment is that of synchronous rolling, which occurs when the time of rolling of the ship is the same as the time of the wave. Inspection

of equations (10) and (11) shows that in this case b is equal to kc, and therefore the general equation (9) may be written

$$\frac{d^2\psi}{dt^2}+b^2\psi=b^2\Phi \sin bt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which can be readily integrated.

The general integral equation takes the form

$$\psi=\alpha \cos bt+\beta \sin bt, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where α and β are arbitrary functions of t. Differentiating twice,

$$\frac{d\psi}{dt}=\frac{d\alpha}{dt} \cos bt-\alpha b \sin bt+\frac{d\beta}{dt} \sin bt+\beta b \cos bt; \quad . \quad . \quad (17)$$

$$\frac{d^2\psi}{dt^2}=\frac{d^2\alpha}{dt^2}\cos bt-\frac{d\alpha}{dt}b \sin bt-\frac{d\alpha}{dt} b \sin bt-\alpha b^2 \cos bt+\frac{d^2\beta}{dt^2} \sin bt$$
$$+\frac{d\beta}{dt} b \cos bt+\frac{d\beta}{dt}b \cos bt-\beta b^2 \sin bt. \quad (18)$$

Add to the above the equation following,

$$b^2\psi=b^2\alpha \cos bt+b^2\beta \sin bt,$$

member to member, and we have

$$\frac{d^2\psi}{dt^2}+b^2\psi=\frac{d^2\alpha}{dt^2} \cos bt-2\frac{d\alpha}{dt}b \sin bt+\frac{d^2\beta}{dt^2} \sin bt+2\frac{d\beta}{dt}b \cos bt. \quad (19)$$

Comparing this equation with the original equation (15), it is apparent that a solution can be obtained if we make,

$$\frac{d^2\alpha}{dt^2}=0 \quad \text{and} \quad \frac{d\beta}{dt}=0,$$

which reduces equation (19) to

$$\frac{d^2\psi}{dt^2}+b^2\psi=-2\frac{d\alpha}{dt}b \sin bt. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Making the first differential coefficient of β equal to zero is equivalent to making β itself constant, which will be indicated by replacing

it by B. Again, if equation (20) is a solution of equation (15), we must have

$$\frac{d\alpha}{dt} = -\tfrac{1}{2}b\Phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

so that

$$\alpha = -\tfrac{1}{2}b\Phi t + A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

where A is another arbitrary constant. Replacing α and β in equation (16) by their equivalents, we have

$$\psi = (A - \tfrac{1}{2}b\Phi t) \cos bt + B \sin bt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

for the angle which the masts of the ship make with the vertical. The angular velocity is

$$\frac{d\psi}{dt} = -b(A - \tfrac{1}{2}b\Phi t) \sin bt + b(B - \tfrac{1}{2}\Phi) \cos bt. \quad . \quad . \quad . \qquad (24)$$

If t is made equal to zero in equations (23) and (24), we shall get for the angle of the masts and for the angular velocity at the origin of time

$$\psi_0 = A$$

and

$$\left(\frac{d\psi}{dt_0}\right) = b(B - \tfrac{1}{2}\Phi),$$

so that the arbitrary constants may take the forms

$$A = \psi_0,$$
$$B = \left\{ \frac{1}{b}\left(\frac{d\psi}{dt_0}\right) + \tfrac{1}{2}\Phi \right\}.$$

Substituting in equations (23) and (24),

$$\psi = (\psi_0 - \tfrac{1}{2}b\Phi t) \cos bt + \left\{ \frac{1}{b}\left(\frac{d\psi}{dt_0}\right) + \tfrac{1}{2}\Phi \right\} \sin bt \quad . \quad . \qquad (25)$$

and

$$\frac{d\psi}{dt} = -b(\psi_0 - \tfrac{1}{2}b\Phi t) \sin bt + \left(\frac{d\psi}{dt_0}\right) \cos kt. \quad . \quad . \quad . \qquad (26)$$

As has already been said, it is convenient to place the ship at the origin, which reduces equation (1), p. 319, to

$$z = -r \cos kct$$

at the beginning of time when t is zero,

$$z = -r,$$

which shows that there is a hollow at the origin in which the ship is assumed to be placed. If it may further be assumed that the ship at the beginning of time is erect and at rest, we shall have

$$\psi_0 = 0 \quad \text{and} \quad \left(\frac{d\psi}{dt_0}\right) = 0.$$

These values when inserted in equations (25) and (26) reduce them to

$$\psi = \tfrac{1}{2}\Phi(\sin bt - bt \cos bt), \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\frac{d\psi}{dt} = \tfrac{1}{2}b^2t\Phi \sin bt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The ship reaches the end of a roll and comes to rest instantaneously when the angular velocity is zero; equation (28) shows that this happens first when $t_1 = \frac{\pi}{b}$. Putting this value in equation (27) gives for the corresponding angle

$$\psi_1 = \tfrac{1}{2}\pi\Phi = \tfrac{1}{2}\pi kr; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

the transformation being by aid of equation (4), p. 320. It is convenient to further transform this expression by replacing k by its value $2\pi \div L$, giving

$$\psi_1 = \frac{\pi^2 r}{L} = 10\frac{r}{L} \text{ nearly}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where L is the length of the wave and r is the amplitude or half the height from hollow to crest.

Inspection of equations (28) and (27) shows that the ship comes

to the end of a roll at the times given below, and has at those times the corresponding inclinations:

$$t_1 = \frac{\pi}{b}, \quad t_2 = 2\frac{\pi}{b}, \quad t_3 = 3\frac{\pi}{b}, \quad t_4 = 4\frac{\pi}{b}, \quad \text{etc.};$$

$$\psi_1 = \psi, \quad \psi_2 = -2\psi_1, \quad \psi_3 = 3\psi_1, \quad \psi_4 = -4\psi_1, \quad \text{etc.}$$

This action may be conventionally represented by Fig. 147, which shows a wave profile running toward the right. If a ship is placed in the hollow of a wave with its masts erect, it will have an incli-

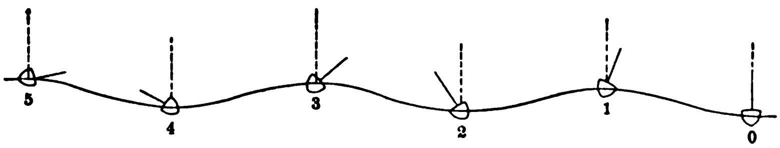


FIG. 147.

nation ψ_1 to the right when the first crest comes to it; the next hollow will give it an inclination $2\psi_1$ to the left; the second crest will give it an inclination $3\psi_1$ to the right, and so on.

Suppose that the height of a wave which synchronizes with the rolling of a ship is two hundredths of the length so that the amplitude r is one hundredth of the length L, then by equation (30)

$$\psi_1 = 10\frac{r}{L} = 0.1 \text{ (circular measure)},$$

or

$$\psi_1 = 5\tfrac{1}{2} \text{ degrees.}$$

The inclination at the end of a roll after the passage of an entire wave will be eleven degrees, and it will take only thirty-three waves to entirely overturn the ship. This conclusion is indeed a strained and unwarranted extension of a theory that is confessedly incomplete, but it is found that a ship rolling in waves that nearly synchronize with the time of rolling will quickly attain very large inclinations. Usually the time of rolling of the ship is not strictly isochronous, and the lengths and times of the waves vary somewhat; and further, the synchronism is seldom exact even for the mean time of the waves. Mr. Wm. Froude, by using small cylindrical models

with very little stability, succeeded in capsizing them by the action of waves having the same period.

General Case.—If the time of rolling of the ship is not synchronous with the time of oscillation of the waves, and is neither very slow nor very quick compared with the time of the waves, we must return to the general equation

$$\frac{d_2\psi}{dt^2}+b^2\psi=b^2\Phi \sin kct.$$

The general solution for this case is

$$b\psi=\sin bt\int\Phi b^2 \sin kct\,.\cos bt\,dt-\cos bt\int\Phi b^2 \sin kct\,.\sin bt\,dt$$
$$+bB\sin bt+bA\cos bt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To transform this equation it is convenient to use the following well-known trigonometrical relations of any two angles, such as C and D:

$$\left.\begin{aligned}2\sin\frac{C+D}{2}\cos\frac{C-D}{2}&=\sin C+\sin D,\\ 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}&=\cos D-\cos C.\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

To apply these equations to the case in hand we may take

$$\frac{C+D}{2}=kc \quad\text{and}\quad \frac{C-D}{2}=b,$$

so that

$$C=kc+b, \quad D=kc-b.$$

Making the application to equation (1) gives

$$\begin{aligned}b\psi=\tfrac{1}{2}\sin bt\,.\,b^2\Phi&\int\{\sin(kc+b)t+\sin(kc-b)t\}dt\\ -\tfrac{1}{2}\cos bt\,.\,b^2\Phi&\int\{\cos(kc-b)t-\cos(kc+b)t\}dt\\ +bB\sin bt+bA\cos bt;&\end{aligned}$$

$$\therefore b\psi = -\tfrac{1}{2}\sin bt \,.\, b^2\Phi\left\{\frac{\cos(kc+b)t}{kc+b} + \frac{\cos(kc-b)t}{kc-b}\right\}$$
$$-\tfrac{1}{2}\cos bt \,.\, b^2\Phi\left\{\frac{\sin(kc-b)t}{kc-b} - \frac{\sin(kc+b)t}{kc+b}\right\}$$
$$+bB\sin bt + bA\cos bt;$$

$$\therefore b\psi = \frac{1}{2}\cdot\frac{b^2\Phi}{b^2-k^2c^2}\{(kc-b)(\sin bt\cos bt\cos kct$$
$$-\sin^2 bt\sin kct - \sin bt\cos bt\cos kct - \cos^2 bt\sin kct)$$
$$+(kc+b)(\sin bt\cos bt\cos kct + \sin^2 bt\sin kct$$
$$-\sin bt\cos bt\cos kct + \cos^2 bt\sin kct)\}$$
$$+bA\cos bt + bB\sin bt.$$

$$\therefore \psi = A\cos bt + B\sin bt + \frac{b^2\Phi}{b^2-k^2c^2}\sin kct. \quad . \quad .$$

Differentiating,

$$\frac{d\psi}{dt} = -Ab\sin bt + Bb\cos bt + \frac{b^2\Phi kc}{b^2-k^2c^2}\cos kct. \quad .$$

At the beginning of time when $t=0$, equation (3) gives

$$\psi_0 = A,$$

and equation (4) gives

$$\left(\frac{d\psi}{dt_0}\right) = Bb + \frac{b^2kc\Phi}{b^2-k^2c^2}.$$

$$\therefore \psi = \psi_0\cos bt + \frac{1}{b}\left\{\left(\frac{d\psi}{dt_0}\right) - \frac{b^2kc\Phi}{b^2-k^2c^2}\right\}\sin bt + \frac{b^2\Phi}{b^2-k^2c^2}\sin k$$

and

$$\frac{d\psi}{dt} = -\psi_0 b\sin kt + \left\{\left(\frac{d\psi}{dt_0}\right) - \frac{b^2kc\Phi}{b^2-k^2c^2}\right\}\cos bt + \frac{b^2kc\Phi}{b^2-k^2c^2}\cos k$$

As in the case of synchronous rolling assume that the placed at the origin in the hollow of a wave with the mast e that

$$\psi_0 = 0 \quad \left(\frac{d\psi}{dt_0}\right) = 0;$$

then

$$\psi = \frac{\Phi b^2}{b^2-k^2c^2}\sin kct - \frac{kc}{b}\,\frac{\Phi b^2}{b^2-k^2c^2}\sin bt. \quad . \quad . \quad .$$

It is evident that the inclination of the ship is the resultant of two angular oscillations which may be represented by

$$\psi_1=\frac{\Phi b^2}{b^2-k^2c^2}\sin kct=\Psi_1\sin kct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\psi_2=\frac{kc}{b}\frac{\Phi b^2}{b^2-k^2c^2}\sin bt=\Psi_2\sin bt=\frac{kc}{b}\Psi_1\sin bt. \quad . \quad . \quad (9)$$

The maximum amplitudes of the individual oscillations are

$$\Psi_1=\frac{\Phi b^2}{b^2-k^2c^2} \quad \text{and} \quad \Psi_2=\frac{kc}{b}\Psi_1=\frac{kc}{b}\frac{\Phi b^2}{b^2-k^2c^2}. \quad . \quad . \quad . \quad (10)$$

The time required for half an oscillation is obtained by making $\psi_1=\Psi_1$ and $\psi_2=\Psi_2$, which gives

$$\sin kct=1, \qquad \sin bt=1,$$

or

$$kct=\sin^{-1}1=\frac{\pi}{2}, \qquad bt=\sin^{-1}1=\frac{\pi}{2}.$$

Consequently the time for a single component oscillation from side to side is

$$t_1=\frac{\pi}{kc}=\tfrac{1}{2}T, \quad \text{or} \quad t_2=\frac{\pi}{b}=t. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The first component oscillation is consequently synchronous with the oscillation of the waves, and the second oscillation is synchronous with the unresisted rolling of the ship in quiet water. The ratio of the maximum inclinations is

$$\Psi_1:\Psi_2=\frac{1}{kc}:\frac{1}{b}=t_1:t_2.$$

The times at which the ship reaches its maximum inclination can be found by replacing ψ_0 and $\left(\frac{d\psi}{dt_0}\right)$ by zero in the equation (6) for the first differential coefficient, and then equating that coefficient to zero; this gives

$$\cos kct=\cos bt.$$

$$\therefore\ b\phi = -\tfrac{1}{2}\sin bt\,.\,b^2\Phi\left\{\frac{\cos(kc-b)t}{kc-b} - \frac{\cos(kc-b)t}{kc-b}\right\}$$

$$-\tfrac{1}{2}\cos bt\,.\,b^2\Phi\left\{\frac{\sin(kc-b)t}{kc-b} - \frac{\sin(kc+b)t}{kc+b}\right\}$$

$$+bB\sin bt + bA\cos bt;$$

$$\therefore\ b\phi = \frac{1}{2}\cdot\frac{b^2\Phi}{b^2-k^2c^2}\{(kc-b)(\sin bt\cos bt\cos kct$$

$$-\sin^2 bt\sin kct - \sin bt\cos bt\cos kct - \cos^2 bt\sin kct)$$

$$+(kc+b)(\sin bt\cos bt\cos kct + \sin^2 bt\sin kct$$

$$-\sin bt\cos bt\cos kct + \cos^2 bt\sin kct)\}$$

$$+bA\cos bt + bB\sin bt.$$

$$\therefore\ \phi = A\cos bt + B\sin bt + \frac{b^2\Phi}{b^2-k^2c^2}\sin kct. \quad . \quad . \quad . \qquad (3)$$

Differentiating,

$$\frac{d\psi}{dt} = -Ab\sin bt + Bb\cos bt + \frac{b^2\Phi kc}{b^2-k^2c^2}\cos kct. \quad . \quad . \qquad (4)$$

At the beginning of time when $t=0$, equation (3) gives

$$\psi_0 = A,$$

and equation (4) gives

$$\left(\frac{d\psi}{dt_0}\right) = Bb + \frac{b^2kc\Phi}{b^2-k^2c^2}.$$

$$\therefore\ \psi = \psi_0\cos bt + \frac{1}{b}\left\{\left(\frac{d\psi}{dt_0}\right) - \frac{b^2kc\Phi}{b^2-k^2c^2}\right\}\sin bt + \frac{b^2\Phi}{b^2-k^2c^2}\sin kct \qquad (5)$$

and

$$\frac{d\psi}{dt} = -\psi_0 b\sin kt + \left\{\left(\frac{d\psi}{dt_0}\right) - \frac{b^2kc\Phi}{b^2-k^2c^2}\right\}\cos bt + \frac{b^2kc\Phi}{b^2-k^2c^2}\cos kct. \qquad (6)$$

As in the case of synchronous rolling assume that the ship is placed at the origin in the hollow of a wave with the mast erect, so that

$$\psi_0 = 0 \quad \left(\frac{d\psi}{dt_0}\right) = 0;$$

then

$$\phi = \frac{\Phi b^2}{b^2-k^2c^2}\sin kct - \frac{kc}{b}\,\frac{\Phi b^2}{b^2-k^2c^2}\sin bt. \quad . \quad . \quad . \quad . \qquad (7)$$

It is evident that the inclination of the ship is the resultant of two angular oscillations which may be represented by

$$\psi_1 = \frac{\Phi b^2}{b^2 - k^2c^2} \sin kct = \Psi_1 \sin kct \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\psi_2 = \frac{kc}{b} \frac{\Phi b^2}{b^2 - k^2c^2} \sin bt = \Psi_2 \sin bt = \frac{kc}{b} \Psi_1 \sin bt. \quad . \quad . \quad (9)$$

The maximum amplitudes of the individual oscillations are

$$\Psi_1 = \frac{\Phi b^2}{b^2 - k^2c^2} \text{ and } \Psi_2 = \frac{kc}{b} \Psi_1 = \frac{kc}{b} \frac{\Phi b^2}{b^2 - k^2c^2}. \quad . \quad . \quad . \quad (10)$$

The time required for half an oscillation is obtained by making $\psi_1 = \Psi_1$ and $\psi_2 = \Psi_2$, which gives

$$\sin kct = 1, \qquad \sin bt = 1,$$

or

$$kct = \sin^{-1} 1 = \frac{\pi}{2}, \qquad bt = \sin^{-1} 1 = \frac{\pi}{2}.$$

Consequently the time for a single component oscillation from side to side is

$$t_1 = \frac{\pi}{kc} = \tfrac{1}{2}T, \text{ or } t_2 = \frac{\pi}{b} = t. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The first component oscillation is consequently synchronous with the oscillation of the waves, and the second oscillation is synchronous with the unresisted rolling of the ship in quiet water. The ratio of the maximum inclinations is

$$\Psi_1 : \Psi_2 = \frac{1}{kc} : \frac{1}{b} = t_1 : t_2.$$

The times at which the ship reaches its maximum inclination can be found by replacing ψ_0 and $\left(\frac{d\psi}{dt_0}\right)$ by zero in the equation (6) for the first differential coefficient, and then equating that coefficient to zero; this gives

$$\cos kct = \cos bt.$$

The two solutions of this equation are

$$bt' = -kct' + 2\pi n, \therefore t' = \frac{2\pi}{b+kc}n; \quad . \quad . \quad . \quad . \quad (12)$$

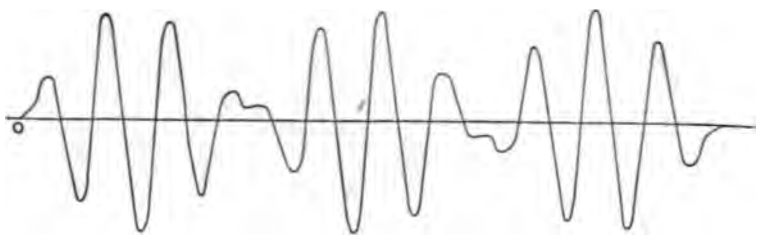
and

$$bt'' = kct'' + 2\pi n, \therefore t'' = \frac{2\pi}{b-kc}n. \quad . \quad . \quad . \quad . \quad (13)$$

The intervals of time between two successive single oscillations (or rolls of the ship from side to side) under this condition are

$$t_0' = \frac{2\pi}{b+kc} \text{ and } t_0'' = \frac{2\pi}{b-kc}. \quad . \quad . \quad (14) \text{ and } (15)$$

FIG. 148.



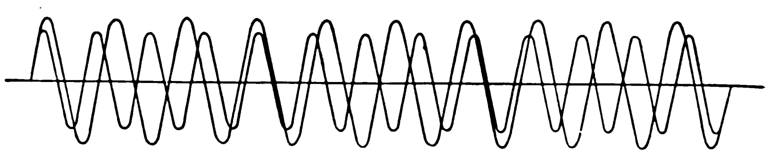


FIG. 149.

The problem may be conveniently represented graphically, as in Fig. 149, where the abscissa represent times, and the ordinates represent inclinations of the ship from the absolute vertical. In this case it is assumed that the ship in quiet water would make thirteen rolls in the time required for the passage of ten waves, or that

$$\frac{kc}{b} = \frac{10}{13}.$$

The first term of the value of ψ (see equation (7)), namely,

$$\frac{\Phi b^2}{b^2 - k^2c^2} \sin kct,$$

can be represented by a sinusoid having ten crests and ten hollows, as represented in Fig. 149. The second term, namely,

$$\frac{kc}{b} \frac{\Phi b^2}{b^2 - k^2c^2} \sin bt,$$

can be represented by a sinusoid having thirteen crests and thirteen hollows. The difference of the ordinates at any point on the horizontal axis represents the absolute inclination of the ship at the corresponding instant of time. Fig. 148 is plotted with these differences, and represents the oscillations of the ship beginning at rest in one hollow of a wave and extending till it is again erect in the tenth succeeding hollow; the pattern is repeated for each ten waves.

The interval between two maxima by equation (14) is

$$t_0' = \frac{2\pi}{b + kc} = \frac{10}{13 + 10} \cdot \frac{2\pi}{kc} = \frac{10}{23} \frac{2\pi}{kc},$$

which, compared with the time of oscillation of the waves,

$$T = \frac{2\pi}{kc},$$

shows that there are twenty-three such intervals for ten oscillations of the waves. The interval between two maxima by equation (15) is

$$t_0'' = \frac{2\pi}{b - kc} = \frac{10}{13 - 10} \frac{2\pi}{kc} = \frac{10}{3} \frac{2\pi}{kc},$$

so that there are three such intervals for ten oscillations of the waves.

Starting at the origin O, there is first a half-roll to the right in the time t_0', then three rolls to the left, each followed by a roll to the right; making in all seven such intervals. We now come to the end of the first interval t_0'', and the ship makes a curious movement toward the left, recovers partially, and then goes on for seven more of the short intervals, after which we come to the end of the second long interval, at which the ship reverses this peculiar motion, and then in seven more short intervals completes the first

series of rolls corresponding to ten waves. This discussion throws some light on the erratic rolling of a ship at sea, where further complication is introduced by the facts that the waves are not all the same length and the rolling of the ship in quiet water is not exactly isochronous.

If the time of rolling of a ship and the time of oscillation of the waves are incommensurable, then the ship will never come again to exactly the same relative position with regard to a wave, and successive series of rolls will not be alike.

An interesting case arises when the denominator of the fraction $\frac{kc}{b}$, reduced to its lowest terms, is one unit larger or smaller than the numerator. The long period becomes then equal to the time of a complete series of rolls, and the ship then has isochronous rolling if we consider that the first roll of a series starting from rest in the hollow of a wave is a complete roll from zero at the left to the first maximum to the right. For example, if

$$\frac{kc}{b} = \frac{10}{11},$$

then the short interval is

$$t_a = \frac{2\pi}{b + kc} = \frac{10}{11 + 10} \cdot \frac{2\pi}{kc} = \frac{10}{21} \frac{2\pi}{kc},$$

so that there will be twenty-one rolls for each ten waves. The rolls of a series will gradually increase to the middle of the series and then will decrease to zero at the end of the series.

As an example consider the case of a ship rolling among waves 1000 feet long and having a height of 10 feet. Then by equation (4), page 320,

$$\sin \Phi = kr = \frac{r}{L \div 2\pi} = \frac{10 \times 2\pi}{1000},$$

or approximately

$$\Phi = \frac{\pi}{50} = \frac{180^\circ}{50} = 3\tfrac{1}{2}^\circ.$$

The greatest inclination which the ship will take is, by equation (7),

$$\Phi\left(\frac{b^2}{b^2-k^2c^2}+\frac{kc}{b}\frac{b^2}{b^2-k^2c^2}\right),$$

so that if $b=\frac{13}{10}kc$, this greatest inclination is

$$3\tfrac{1}{2}\left(\frac{13^2}{13^2-10^2}+\frac{10}{13}\cdot\frac{13^2}{13^2-10^2}\right)=15^\circ.$$

If $b=\frac{11}{10}kc$, the greatest inclination will be nearly

$$3\tfrac{1}{2}\left(\frac{11^2}{11^2-10^2}+\frac{10}{11}\cdot\frac{11^2}{11^2-10^2}\right)=40^\circ.$$

The last case shows for eleven rolls of the ship an accumulated inclination of 40°. Compare this result with the computation for synchronous rolling under the same conditions on page 326, where it appeared that each wave added 11° to the maximum inclination of the ship; from this it appears that with synchronous rolling the passage of four waves would give an inclination of 44°, and that further the successive waves would keep on adding to the inclination at the same rate. As will be seen later, resisted rolling of actual ships at sea, whether synchronous or not, is unlikely to attain so large angles.

From the table on page 261 it appears that a wave 1000 feet long has a time of 14 seconds. If a ship makes a double roll in $\frac{10}{11}$ of the time of this wave, then a single roll of the ship from side to side will require $5\frac{1}{2}$ seconds; large steamships commonly have a time of rolling of seven or eight seconds.

Effective Wave and Effective Wave Slope.—In the theory of the rolling of a ship among waves just developed, it is assumed that the ship is very small compared to the waves, and that it does not affect the form of the waves. Actual ships are far from conforming to these conditions, and in particular the draught of the ship is always considerable compared with the height of waves at sea. Now the radius of the orbits of particles of a wave decreases rapidly with the depth, as is seen from the table on page 262, and consequently in

investigations of the rolling of actual ships among waves some allowance must be made for the diminution of the wave disturbance with the draught of the ship.

Mr. William Froude proposed for a working assumption that the rolling of a ship among waves should be referred to the wave surface which passes through the centre of buoyancy of the ship, and that this wave surface be called the effective wave surface. The maximum slope of this wave surface he called the effective wave slope, and he used it for Φ in his equations for rolling among waves. Investigations of the rolling of ships among waves by the aid of proper registering apparatus shows a fair conformity of the theory with the behavior of the ships, when these assumptions are made.

Pitching among Waves.—The study of pitching among waves is less satisfactory than that of rolling because large ships are long when compared with any waves they are likely to meet; very commonly a long ship will lie across two or more waves. On the other hand, the time of pitching of a ship is likely to be about half as much as the time of rolling; large ships may have three or four seconds for the time of pitching. A double oscillation of this nature may therefore take six to eight seconds, which corresponds to the times of waves that are 200 to 350 feet long. A ship which meets with waves that have an effective time nearly coinciding with the time of pitching will be likely to pitch heavily, and such pitching may be even more distressing than rolling to those who are unaccustomed to the sea. Heavy pitching is likely to throw the propeller of a ship partly out of water and make the engine race; to check the racing it is necessary to shut steam off from the engine momentarily, and if this happens frequently, the ship loses speed appreciably. In some cases the pounding of a ship that is steaming into a head sea may be so distressing that the speed is reduced. A ship that has twin screws will have them thrown up both by rolling and by pitching, and may be more liable to have the engines race than a ship with a single propeller.

Effective Time of Waves.—When a ship is steaming at an angle with the crests of the waves, the effective time of the waves is that between the arrival of succeeding crests, and will depend on the direction and speed of the ship. However long may be the period

of rolling of a ship, there will be some combination of direction and speed that will make the effective time of the waves nearly synchronize with the time of rolling of the ship, and under such conditions the ship may be expected to roll heavily.

In like manner a ship which is steaming directly or obliquely across the crests of waves will have its pitching affected by the effective time of the waves, and, especially when steaming against a head sea, is likely to produce a synchronism that may induce heavy pitching.

Heaving among Waves.—The time of heaving of a ship in quiet water is so short that it is not likely that any wave which can affect a ship will synchronize with it.

There remains, however, the consideration of the effect of waves on the apparent weight of the ship, by which is meant the mass of the ship multiplied by the total acceleration acting on it, just as the weight in quiet water is the mass multiplied by the acceleration due to gravity.

Equation (10) on page 254 gives for the vertical acceleration of a particle of water on account of wave action

$$\frac{d^2z}{dt^2} = c^2ke^{kb} \cos k(a+ct).$$

Replacing c, k, and e^{kb} by values found on pages 255, 256, and 259, and applying the equation to the hollow of a wave at the origin and at the beginning of the time, gives for the acceleration due to wave action

$$\frac{2\pi r}{L}g.$$

This is to be added to the acceleration due to gravity, giving for the total acceleration at the hollow of a wave

$$g\left(1+\frac{2\pi r}{L}\right). \quad \dots \dots \dots \quad (1)$$

In much the same way it may be shown that the acceleration at the crest of a wave is

$$g\left(1-\frac{2\pi r}{L}\right). \quad \dots \dots \dots \quad (2)$$

For example, the apparent weight of a ship on the crest of a wave 2000 feet long and 40 feet high is 0.94 of the displacement in quiet water, and the apparent weight in a hollow is 1.06 of that displacement.

The principal interest of this discussion lies in its application to stability of ships when rolling among waves. For steamships with little or no sail the variations of total acceleration, apparent weight, and righting moment do not appear to be of much consequence, because the waves which cause the rolling are affected by the same variation of acceleration, and their ability to produce inclination varies as does the righting moment of the ship. The case is quite different for sailing-ships, which are inclined under the pressure of the wind, which has an inclining moment that is independent of the waves. A ship may therefore be expected to heel over farther under wind-pressure when on the crest of a wave than when in a hollow, but it is likely that any such effect will be confused with the rolling due to the action of the waves. It is said that small boats are more likely to capsize on the crest of a wave, but it may be that the boat is somewhat shielded from the wind when in a hollow.

Apparent Weight.—When the ship is rolling, any body on the ship, or any portion of the structure of the ship, is subject, at any instant, to two forces: (1) the attraction of gravity, and (2) the force which imparts to it the acceleration due to rolling. The resultant of these two forces is known as the *apparent weight.*

In order to find the acceleration due to rolling at a given point on a ship, it is necessary to know its distance from the axis about which the ship rolls, the time of rolling, and the extent of rolling. Unresisted rolling takes place about an axis through the centre of gravity of the ship; rolling in a resisting medium, as will appear later, is sensibly an oscillation about an axis somewhat higher than the centre of gravity.

For the present purpose it is sufficient to deal with simple isochronous rolling, for which, by equation (3), page 311, the acceleration at a given inclination is

$$\frac{d^2\theta}{dt^2} = -b^2\theta.$$

If the given point is at the distance l from the axis of rolling, then the tangential acceleration is

$$j_t = l\frac{d^2\theta}{dt^2} = -lb^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The angular velocity by equation (5), page 311, is

$$\frac{d\theta}{dt} = b\sqrt{\theta_m{}^2 - \theta^2},$$

so that at the distance l from the axis of rolling there is a radial acceleration

$$j_r = l\left(\frac{d\theta}{dt}\right)^2 = lb^2(\theta_m{}^2 - \theta^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In these equations b has the value assigned to it on page 321. It is, however, more convenient to derive b from equation (6), page 311, which gives

$$b = \frac{\pi}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The maximum tangential acceleration is found at the end of a roll, where the inclination is θ_m. The maximum radial acceleration is found at the middle of a roll, where the inclination is zero, i.e., when the ship is erect. The maxima are

$$J_t = -lb^2\theta_m = -\frac{\pi^2}{t^2}l\theta_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$J_r = lb^2\theta_m{}^2 = \frac{\pi^2}{t^2}l\theta_m{}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The ratio of the maxima (neglecting the sign of J_t) is

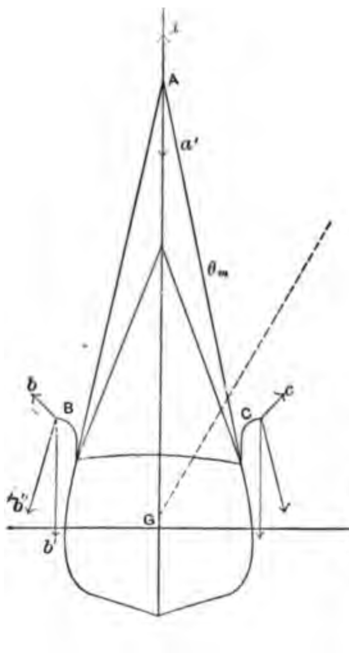
$$\frac{J_t}{J_r} = \frac{1}{\theta_m}.$$

Now θ_m is seldom, if ever, equal to unity (57°), so that the tangential acceleration is always the larger. For a roll of 30° ($\theta_m = 0.52$) J_r is nearly twice J_t.

The negative sign attached to the value for j_t or J_t shows that the force exerted by the body at a given point is directed away from the

direction of motion. Thus in Fig. 151, which represents the ship at the end of the roll, the force Aa at the masthead is directed toward the right, while the ship moves toward the left on the next roll.

At the end of a roll the radial force is zero, and the force due to J_t is at right angles to a line through the axis of rolling. Thus Aa is perpendicular to the mast, and Bb and Cc are perpendicular to



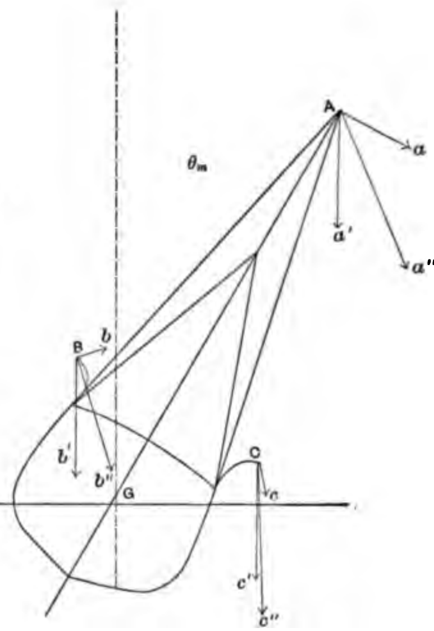


FIG. 150. FIG. 151.

lines that may be drawn from G to B and from G to C. The apparent weight at the masthead is Aa'', the resultant of Aa and of Aa', the attraction of gravity.

The force at C very nearly coincides with the attraction of gravity; for an approximation it may be simply added to get the resultant.

For example, suppose that $OB = l = 30$ ft., that $\theta_m = 30° = 0.52$, and that $t = 8''$; then

$$lb^2 = \frac{\pi^2 l}{t^2} = \frac{\overline{3.1416}^2 \times 30}{8^2} = 4.65,$$

and $J_t = 2.4$, which is not important compared with $g = 32.2$. If the ship rolled twice as quickly, then the force would be four times as large, or 9.6, which is appreciable; but a ship so large as to make $l = 30$ is not likely to roll so quickly, and 30° is an extraordinarily heavy roll.

When the ship is at the middle of its roll, Fig. 150, the tangential force is zero, but the radial force is then a maximum. For a roll of 30° J_r is half of J_t, or $J_r = 1.2$ at B. For the point A at the masthead for a value of $l = 100$ feet

$$J_t = -\frac{\pi^2}{t^2} 100 \times 0.52 = 8,$$

which is about $\frac{1}{4}$ of $g = 32.2$. The value of J_t at the masthead, which is twice J_r for a roll of 30°, is a very important matter. More properly, we should deal with the mast and the attached spars, and should find the amount of the tangential force and its point of application for each number separately. The spars are commonly at right angles to the mast, and the point of application of the force of acceleration will be at the place where they cross the mast. Each mast, lower mast, topmast, etc., may be treated as a straight stick with the force of acceleration at the radius of gyration about the axis of rolling. The mast is usually stepped on the keel, and is fixed at the deck by mast-partners; it is also stayed by the shrouds and by stays. The calculation becomes somewhat complicated, and is not very satisfactory; the fixing of masts is guided mainly by experience. The stresses from rolling are a serious source of danger, as sailing-ships have been known to roll their masts out; for example, the British 74 *Berwick*, in the wars of the French Revolution, rolled her masts overboard at anchor in San Fiorenzo Bay.

It does not appear that it is possible or necessary to take the effect of rolling into account in designing fittings for ships.

Apparent Weight from Pitching.—Following for pitching a line of reasoning analogous to that for rolling, it appears that the maximum tangential acceleration is given by the equation

$$J_t = -\frac{\pi^2}{T^2} l \theta_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which the letters have the same significance as in equation (4), except that T, the time of pitching, takes the place of t, the time of rolling. The effect of pitching near the ends of a ship may be very notable. Thus if the length of a ship is 600 feet, if the maximum deviation is $3^\circ = \theta_m = 0.0507$, and if the time of pitching is 4 seconds, then

$$J_t = \frac{\overline{3.1416}^2}{4^2} \times 300 \times 0.0507 = 9.4,$$

which is more than $\frac{1}{4}$ of $g = 32.2$. The bow of the ship in such case would rise and fall twice this amount, or 18.8 feet.

Methods of Measuring Rolling.—If the horizon is visible, the

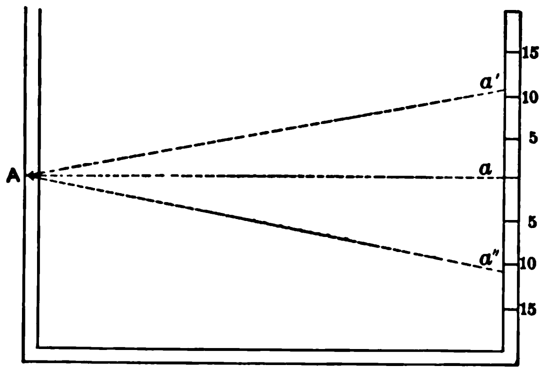


FIG. 152.

duration and amplitude of successive rolls can be readily observed by aid of a simple instrument represented by Fig. 152.

At A there is a horizontal slit, and at $a'a''$ there is a scale graduated to give degrees. An observer places his eye at A and notes the angle over which the horizon appears to move from a' to a''; a second observer with a watch or chronometer notes the time required for the successive oscillations. The results may be plotted with times as abscissæ and angles as ordinates, as in Fig. 153, which represents the rolling of a ship which is artificially inclined 10° from the vertical and then allowed to roll until it comes to rest. If the ship is at first inclined to the right, the ordinates at 1, 3, 5, etc., represent inclinations to the left, while 2, 4, 6, etc., represent inclinations to the right; it is convenient for the present purpose to ignore the direc-

tion of inclination and lay off all ordinates above the axis as shown. A curve drawn through the point *a*, which represents the original inclination, and the points *b*, *c*, *d*, etc., which represent the inclinations at the ends of the successive rolls to left and right, is called the curve of decrease of rolling. A photographic record of rolling can

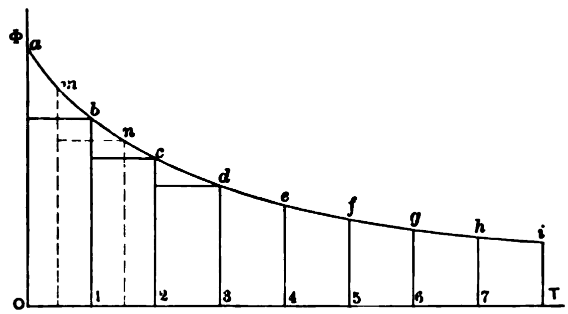


FIG. 153.

be obtained by aid of a camera fixed athwartship on the bridge or other position where an uninterrupted view of the horizon can be had. In place of the usual plate-holder there may be a clockwork for drawing a sensitized film along at a determined rate, so that times are readily determined. In front of the lens is a vertical slot through which a narrow image of the sea and sky divided by the horizon is focussed on the sensitive film. As the ship rolls the horizon-line swings up and down over the film and a continuous series of photographs are taken, so that when the film is developed it shows a sinuous line from which the extent and the nature of the rolling can be determined. To determine the angle of a roll, a mean line can be drawn between the crests and hollows of the horizon-curve, when the ordinate to the curve can be measured and divided by the distance of the film from the lens, thereby obtaining the tangent of the angle.

Long and Short Pendulums.—Any pendulum on shipboard will tend to take the direction of the apparent force. If the ship is at rest in a dock, the apparent force is the attraction of gravity. If the ship is rolling in quiet water, the apparent force is the resultant of the forces producing the acceleration due to rolling compounded with the attraction of gravity. If the ship is rolling among waves, the apparent force is affected both by the motion of the waves and the

rolling of the ship. At any instant the apparent force at a given point on the ship is the resultant of the force due to acceleration as affected both by the waves and by rolling, and of the attraction of gravity.

A short pendulum which has no proper oscillation of its own will very nearly indicate the direction of the apparent force, since it responds quickly to the forces acting on it. A long pendulum will move so slowly that it practically remains vertical, unless it has an oscillation of its own.

Now a pendulum beating seconds is about 39 inches long (varying with the latitude and elevation above the sea), and the time of oscillation is proportional to the square root of its length. Since the time of rolling of ships varies from four to eight seconds, the second pendulum will answer for showing the direction of apparent weight; but a shorter pendulum will be better and more convenient. The pendulum often supplied for this purpose is about 20 inches long and beats in 0.7 of a second. For careful investigations, such as those made by the Froudes and by Bertin, a very short pendulum was used. Bertin's short pendulum beat in 0.2 of a second and was about 1.6 inches from point of suspension to point of oscillation. A long, light arm, carrying a pencil, was used to indicate the deviation of the pendulum.

A pendulum that is slow as compared with the time of rolling of a ship would have to be too long if in the form of a common pendulum. The form actually used is a wheel with the centre of gravity about 0.07 of an inch below the axis. The wheel weighs about 400 pounds, and is suspended on friction wheels. The time of oscillation is about 40 seconds.

The short or quick pendulum is often used on ships to indicate the extent of rolling, and for that purpose its indications may be fallacious and misleading. A point at the axis of rolling in still water is at rest, and a short pendulum suspended at that point is affected by gravity only, and indicates the direction of the true vertical, provided it has no oscillation of its own. For most ships the axis of rolling is near the water-line. A point above the axis of rolling, like *A* at the masthead of Fig. 151, page 338, will have the acceleration due to rolling inclined away from the mast, and a short pendulum

at that point will evidently indicate too large an inclination. On the contrary, a short pendulum which is suspended at a point below the axis of rolling will indicate too small an angle of rolling.

Referring to the example on page 339, it appears that the tangential acceleration at the end of a roll is 2.4 feet per second at a point 30 feet from the axis of rolling for a ship which has a time of 8 seconds and is rolling to 30° from the vertical. A short pendulum at such a point would indicate 34° when the angle was really 30°; not a very serious matter. If, however, the ship rolls twice as fast, or has a time of 4 seconds, the acceleration would be 9.6 feet per second and the indication would be 47° instead of 30°.

Since the tangential acceleration is proportional to the distance from the axis of rolling, it appears that a short pendulum at a moderate height (say 10 feet) above the axis of rolling will indicate the correct angle of inclination without an appreciable error when the ship is rolling in still water. If the ship is rolling among waves, the short pendulum under these conditions will indicate the direction of the normal to the effective wave surface. The proper office of the short pendulum is that just indicated, i.e., to show the direction of the normal to the effective wave surface.

The long or slow pendulum, if it has no proper oscillation of its own, will at all times show the direction of the vertical. It must be made in such form as not to offer resistance to moving in air; the usual form of a wheel with eccentric centre of gravity fulfils this condition. This condition, however, carries with it the difficulty that there is no way of quieting the oscillation of the pendulum if it starts to move, and there is no way, when at sea, of recognizing a small oscillation of such a pendulum.

Gyroscope.—Instead of a long pendulum a gyroscope may be used to locate a line fixed in space from which the rolling of the ship can be located. A form of this instrument that was used by Admiral Pâris was in fact a top with a heavy rim hung below the point of the top. This top was spun in the usual way with a string and holder and then deposited on a post supported from a broad base. On shore such a top will take a vertical position whether it is spinning or not, and on shipboard its axis will remain fixed in space when it is spinning; it is, however, liable to have a slow oscillation

of its own much like that of a long pendulum. An obvious improvement of this instrument is to provide an electrical device by which it may be spun continuously. There does not appear to be any advantage in placing the axis vertical, nor in giving the gyroscope any natural stability; in fact the natural stability tends to give the instrument an oscillation of its own which is very undesirable.

Oscillograph.—Investigations of the rolling of ships in quiet water and among waves have been made by Bertin * and by the Froudes,† by aid of instruments known as oscillographs which have slow and quick pendulums, and registering devices. The external forms of the instruments used by the French and English observers differed in external appearance and are too complicated for a proper description at this place; there are, however, certain essential features which will be stated. The short pendulum has a natural time of about 0.2 of a second, and is made wide and flat, so as to offer considerable resistance to the air. Its office is to show the direction of the apparent weight; when the oscillograph is set at a moderate distance above the axis of rolling, the apparent weight is normal to the effective wave surface. The long or slow pendulum is a wheel with eccentric centre of gravity, and has a natural time of about 40 seconds; its office is to show the true vertical. The registering device is a paper drum driven by clockwork at a uniform rate. Three pencils bear against the paper, one from each pendulum, and the third is a fixed pencil which draws the base- or reference-line. This latter is given a small lateral movement by an electro-magnet at uniform intervals of time, usually one second.

Rolling in a Resisting Medium.—When a ship is rolling in a non-resisting medium, the motion consists of a rotation about a horizontal axis through the centre of gravity, together with a vertical translation, up and down, of this centre of gravity due to heaving. If there is no heaving, then there remains only the rotation about a horizontal axis through the centre of gravity. This is evident from the consideration that a horizontal translation of the axis through the centre of gravity can arise only from the action of some

* Théorie du Navire, Pollard and Dudebont.

† Proceedings Inst. Naval Archts., Vol. XIV.

body outside of the ship. The only body with which the ship is in communication is the medium in which it rolls; consequently no horizontal motion can arise so long as the medium is unresisting. To conform to this condition, the medium must not only be frictionless, but must offer no resistance to direct action, as from the pressure on the keel and dead wood.

When a ship rolls in a resisting medium, like water, there will be three kinds of resistance:

1. Resistance of friction of water on the skin of the ship.

2. Resistance due to direct pressure of the keel, dead wood, and bilge-keels (if any).

3. Resistance due to wave-making.

It does not appear possible to determine synthetically a satisfactory form for the general equation for rolling of a ship in still water; but from certain experiments on the extinction of rolling (which will be stated immediately) a provisional form of the general equation has been established which has sufficed to make a comparison of experiments on rolling with the theory and to show a fair concordance between them.

Tranquil Point.—It is clear that a ship rolling in a resisting medium will not roll about an axis through the centre of gravity, for the resistances will cause a translation of the centre of gravity back and forth. There is, however, a region somewhat above the centre of gravity where the motion is very small; this may be called the *tranquil region;* as the region is restricted compared with the cross-section of the ship, it may be called the *tranquil point.* The actual motion of rolling is nearly that of rolling about an axis through the tranquil point. Unless the rolling is very heavy it is very nearly isochronous. It is evident that resisted rolling is not likely to be strictly isochronous even for a ship which has isochronous unresisted rolling, because the forces of resistance, such as friction and direct resistance, vary with the angular velocity; they are zero at the ends of a roll, and are a maximum at or near the middle. On the contrary, the righting moment and the accelerating moment are zero at the middle, and are greatest at the ends of the roll.

Time of Resisted Rolling.—The time of resisted rolling of a ship is appreciably greater than that of unresisted rolling, and the differ-

ence is greater for rolling to large angles than for gentle rolling. It is likely that rolling to small angles is sensibly isochronous and unresisted and may serve as a basis for the determination of the moment of inertia of the ship and for comparison with resisted rolling to large angles.

Curve of Extinction.—Turning to Fig. 153, page 341, it is evident that the difference between two successive ordinates, such as $1b$ and $2c$, is the decrease of inclination for a single roll due to resistance. If preferred, ordinates may be interpolated like those drawn at m and n, and the decrease for a single roll to the angle $1b$ may be determined from them; but refinement is scarcely possible in this investigation.

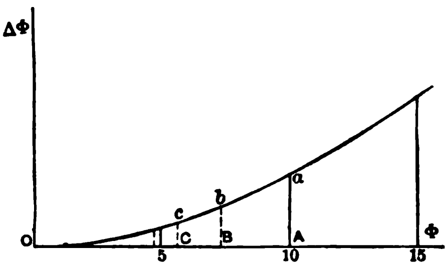


FIG. 154.

In Fig. 154 the decrease of inclination for a single roll is laid off as an ordinate on the total inclination as an abscissa; thus Aa is the decrease for the first single roll from the original inclination of 10°, Bb is the decrease for the second roll, and so on. The curve through the points thus determined is called the curve of extinction of rolling. From it the decrease for a single roll from any angle of inclination is readily determined; it is customary to extend the curve to 15°, so that points can be exterpolated beyond 10°, or whatever angle the ship was originally inclined to. Of course exterpolated ordinates are subject to unknown errors.

It was suggested by Wm. Froude that the curve of extinction for ships can, in general, be represented by an equation having the form

$$\Delta \Psi = A \Psi + B \Psi^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which Ψ represents the angle of inclination at the beginning of a roll, and $\Delta \Psi$ is the loss of amplitude for the next single roll from side to side.

Since there are two arbitrary constants, two observations are required to determine the equation to the curve; or two pairs of coordinates may be taken from the faired curve for the same purpose.

The following table gives values for the constants determined from several ships of the English navy, and from the U. S. S. *Oregon:*

CONSTANTS FOR FROUDE'S EQUATION FOR EXTINCTION OF ROLLING.

	Time of Single Roll.	A	B	Displacement.
Greyhound, without bilge-keels.....	4.35	0.044	0.0032	1160
" with "	4.33	0.035	0.05	
Sultan..........................	....	0.0267	0.00166	9290
Devastation......................	....	0.0720	0.0150	9330
Inconstant.......................	....	0.035	0.0051	
Narcissus........................	....	0.0370	0.0080	5600
Volage...........................	....	0.0280	0.0073	
Revenge, light, no bilge-keels.......	8.0	0.015	0.0028	13370
" deep, " "	7.6	0.0123	0.0025	14300
" light, with bilge-keels.....	8.4	0.0840	0.019	13370
" deep, " "	7.75	0.0650	0.017	14300
Oregon, without bilge-keels........	7.6	0.011	0.0021	9810
" with "	7.83	0.045	0.023	9790

A simple equation is proposed by Bertin, who finds that for inclinations greater than 2°.5 the curve of extinction can be represented by an equation having the form

$$\Delta\Psi = A_0 + B_0\Psi^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Bertin considers that the curve may be continued to the origin from the ordinate at 2°.5 by a straight line. Since this part of the curve of extinction is always doubtful, it has little interest. The table on the next page gives values for the constants for several vessels.

The first constant for the *Suffren* and for the remainder of the ships in the table becomes zero, leading to a further simplification of the equation of extinction. A comparison of ships in either of the tables of constants shows that under like conditions light ships loose their rolling motion more quickly than heavy ships do. For example, the *Sultan*, an armored ship of 9290 tons, when rolling to 20° from the vertical, or 40° in all, loses

$$\Delta\Psi = 0.0267 \times 20 + 0.00166 \times \overline{20}^2 = 1^\circ.2$$

in the first roll. On the other hand, the *Narcissus*, which is a

CONSTANTS FOR BERTIN'S EQUATION FOR EXTINCTION OF ROLLING.

	A_0	B_0
Coal-barge	0.090	0.0154
Barge	0.117	0.0121
Navette	0.061	0.0129
Calvador	0.033	0.0165
Hirondelle	0.013	0.0207
Suffren	0.000	0.0083
Volage	......	0.0141
Inconstant	......	0.0123
Sultan	......	0.0045
Elorn	......	0.0160
Renard	......	0.0124
Eurydice	......	0.0077
La Galliscon[i]ère	......	0.0075

cruiser of 5600 tons, loses under like conditions

$$\Delta \Psi = 0.037 \times 20 + 0.008 \times \overline{20}^2 = 3^\circ.9$$

in the first roll. The reason for this is that the moment of inertia varies as the cube of a linear dimension, while the surface varies as the square of a linear dimension; consequently the stored energy, which must be dissipated as the rolling is extinguished, increases more rapidly than the resistance to rolling, which is mainly superficial. This effect is still more emphasized in comparing a cruiser with an armored ship, which carries large weights of armor on her sides. And further, the smaller ships roll more quickly, and this tends toward a more rapid loss of stored energy.

The same thing may be noticed in comparing the *Revenge* when light and when deep, even though there is greater surface and quicker rolling when deep, both of which tend to extinguish rolling.

Bilge-keels.—The most effective way of extinguishing rolling in quiet water, and of checking rolling among waves, is to provide the ship with bilge-keels. This was shown by Wm. Froude by experiments on the *Greyhound*, a small wooden vessel; the results are shown by the change in the constants in the table. The ship was 172½ feet long, 33¼ feet broad, and drew 13¾ feet; the bilge-keels were 100 feet long and 3½ feet deep, that is, they were excessively large for the ship.

Experiments by Bertin on a barge which had a form somewhat

like a well-shaped ship gave the following results for the constant in the equation

$$\Delta\Psi = B_0 \Psi^2:$$

Without keels	0.0154
With two keels	0.0210
With four keels	0.0293

Similar experiments on the *Elorn* gave:

Without bilge-keels	0.016
With two bilge-keels, immersed	0.030
With two keels at the water-line	0.040

The area of each keel of the barge was about $\frac{1}{28}$ of the area of the water-line. The area of each keel on the *Elorn* was about $\frac{1}{30}$ of the water-line. The keels at the water-line on the *Elorn*, of course, rose out of and struck on the water as she rolled; they were very effective in checking rolling, but such a device on a large ship would be liable to strain her, and the shocks against the water would be very violent.

In June of 1894 very important tests were made on the English battle-ship *Revenge* with and without bilge-keels. Each keel was about 200 feet long and 3 feet deep, tapering toward the ends. The collective area was 1170 square feet, the mean radius from the centre of gravity of the ship was 41 feet when the ship was light and $40\frac{1}{4}$ feet when deep. To show the effect of adding bilge-keels it may be computed that, starting with an inclination of 10°, the ship, when light and without bilge-keels, lost

$$\Delta\Psi = 0.015 \times 10 + 0.0028 \times 10^2 = 0^\circ.43$$

during the first roll; when supplied with bilge-keels the same ship under the same condition lost

$$\Delta\Psi = 0.084 \times 10 + 0.019 \times 10^2 = 2^\circ.7$$

during the first roll. Similar results were obtained from the U. S. S. *Oregon* * when tested with and without bilge-keels. It may be added

* Trans. Soc. Nav. Archts. and Marine Engs., Vol. VI.

that the *Revenge* without bilge-keels was rolled to an angle of 13° by training the heavy barbette guns from side to side at the proper rate. After the bilge-keels were added it was difficult to exceed 6° or 8° by training the guns from side to side, aided by nearly four hundred men running from side to side at the same time. The effect of the bilge-keels was also to increase the time of rolling, as shown in the table.

Bilge-keels, Ship under Way.—The effect of bilge-keels is notably greater when the ship is under way than when lying with no headway, the apparent reason being that the ship continually runs into undisturbed water, which more effectually resists the movement of the keels during rolling.

If the extinction value for the *Revenge* without bilge-keels be taken as 1, then the value without headway and with bilge-keels is 6, at 10 knots speed it is 8.5, and at 12 knots 10.3; these relative values are only for moderate angles up to 6° from the vertical.

Experiments on the *Navette* by Bertin gave for the value of the constant in the equation

$$\Delta\Psi = B_0\Psi^2:$$

Speed	0	4 knots	8 knots.
B_0	0.0109	0.0123	0.015

Quieting Water-chambers.—The common method of inducing rolling on naval vessels, i.e., by having large bodies of men run across the decks in time with the rolling, led to the proposal of chambers partially filled with water and so proportioned that the water should run down the floor of the chamber and be lifted on the next swing upward. It is needless to say that the men must run up the inclined decks to make the ship roll.

Such chambers have been placed on the English central-battery ships *Inflexible* and *Edinburgh*. These ships were given very large metacentric heights, so that some stability would remain even though the unarmored ends should be badly broken up. The great metacentric heights gave quick and heavy rolling, and the addition of water chambers was intended to compensate for this undesirable quality. They were found to be very effective, especially in quieting

small oscillations, which might be very troublesome when the artillery is brought into service.

Swinging Weight.—Mr. Thornycroft* placed on one of his torpedo-boats a device for checking rolling, consisting of a weight that could be shifted laterally by a steam-cylinder, together with an automatic device intended to shift the weight toward the rising side of the boat. Such an automatic device, if controlled by a short pendulum, would tend to keep the boat normal to the wave surface; if a long or slow pendulum were used, it would tend to keep the boat truly vertical, provided the pendulum had no proper oscillation. Though effective on a torpedo-boat, this device is probably not applicable to a large ship.

Use of Small Models.—Mr William Froude, many years ago, made many experiments on small models to discover the probable behavior of ships. The first experiments appear to have been made on a model of the *Great Eastern;* later experiments were made on the *Devastation*, *Inflexible*, and *Edinburgh;* the experiments on the last two included tests on the effect of water-chambers.

Experiments on small models have certain evident advantages, such as small cost and the ease with which experiments may be made for large inclinations. With large ships it is difficult, if not impossible, to produce an inclination of more than 15° for rolling experiments in quiet water.

The use of models for experiments on rolling involves the theory of mechanical similitude which is given on page 410, but the application of that theory to this case is simple and will be stated by itself. In order that the model shall properly represent the ship, it should be so loaded that the model will float at the corresponding depth and will have its radius of gyration and metacentric height proportional to its length. If L, B, D, $R_0 - A$, and P_0 are the length, beam, draught, metacentric height, and radius of gyration of the ship, and if small letters are taken for the corresponding dimensions of the model, then the common ratio is

$$\lambda = \frac{L}{l} = \frac{B}{b} = \frac{D}{d} = \frac{R_0 - A}{r_0 - a} = \frac{P_0}{\rho_0}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Proceedings Inst. Naval Archts., Vol. XXXIII.

The times of rolling of the ship and its model will be, by equation (7), page 312,

$$T=\pi\sqrt{\frac{P_0^2}{g(R_0-A)}} \quad \text{and} \quad t=\pi\sqrt{\frac{\rho_0^2}{g(r_0-a)}}; \quad . \quad . \quad . \quad (2)$$

consequently the ratio of the times is $\sqrt{\lambda}$

If the ship and its model are inclined to the same angle and released, their angular velocities at the same inclination will be inversely proportional to the times of rolling, so that if Ω and ω are the angular velocities,

$$\frac{\Omega}{\omega}=\frac{t}{T}=\frac{1}{\sqrt{\lambda}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the linear velocities at corresponding points on the surfaces in contact with water will have the ratio

$$\frac{L\Omega}{l\omega}=\frac{\lambda}{\sqrt{\lambda}}=\sqrt{\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the discussion of the resistance to the propulsion of a ship through the water, it appears that the resistance can be distinguished to be of three kinds: (1) frictional resistance, (2) direct resistance, and (3) wave-making resistance; and further, from tests on ships and their models and from other experiments we have some knowledge of the nature and methods of variation of these resistances. Though the difference of the conditions of rolling and of propulsion through the water forbid the transfer of constants determined from propulsion to rolling, we may draw certain general conclusions. For instance, it is known that the frictional resistance varies as the surface and nearly as the square of the speed. The surface of a ship and its model vary as the square of a linear dimension (λ^2), and the linear velocities by equation (4) vary as the $\sqrt{\lambda}$ for rolling; consequently the resistance of friction may be considered to vary as

$$\lambda^2\times(\sqrt{\lambda})^2=\lambda^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The moments of the resistances will consequently vary as

$$\lambda^3\,.\,\lambda=\lambda^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The nature of direct resistance is not so well known largely because well-formed ships have little direct resistance; what little is known shows a variation in general like that of friction, premising that the surfaces giving direct resistance are like wide sterns of wooden vessels.

The wave-making resistance of a ship varies as a linear dimension (λ) and as the fourth power of the speed. If this condition is transferred to rolling, the resistance, by aid of equation (4), appears to vary as

$$\lambda \times (\sqrt{\lambda})^4 = \lambda^3,$$

which leads to the same conclusion as the discussion of friction, namely, that the moment varies as the fourth power of a linear dimension (λ^4). But wave-making resistance during rolling differs essentially from that due to the propulsion of a ship through the water, and in particular the wave-making during rolling is directly connected with the direct resistance, which, especially for a ship with bilge-keels, has an important influence in checking rolling. However much legitimate objection to this comparison there may be, it has one good feature in that it directs attention to the importance of wave-making resistance to rolling, especially for quick rolling.

If it be conceded that the resistance to rolling varies as the fourth power of a linear dimension, then it should be expected that good results should be obtained from experiments on models, because the righting moment

$$D(r_0 - a)\theta$$

for small angles very clearly varies as the fourth power of a linear dimension.

Experiments by Mr. R. E. Froude on the rolling of ships and models show a fair correspondence of results. It is undoubtedly safer to consider that tests on models show relative rather than absolute results.

Maximum Resisted Rolling.—The most important result that can be determined from the theory of rolling and the comparison with results of experimental investigations is the maximum angle to which a ship is likely to roll under unfavorable circumstances, such as when placed in the trough of the sea among waves that

synchronize with the time of rolling. It is shown on page 325 that if Φ is the maximum wave slope, then the ship, if it had no resistance to rolling, would accumulate an added inclination of $\frac{1}{2}\pi\Phi$ degrees for each single roll.

If a ship has a known inclination at the end of a given roll, it may be predicted that the angle of inclination at the end of the next succeeding roll will be obtained by adding thereto the increment due to synchronous, unresisted rolling and subtracting the decrement due to resistance. When the increment and decrement are equal the ship may be expected to have reached its maximum rolling, beyond which it will not pass in waves of the given length and height.

Using Froude's equation for extinction of rolling, this conclusion leads to the equation

$$\tfrac{1}{2}\pi\Phi = a\Psi + b\Psi^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which Φ is the effective wave slope and Ψ is the maximum inclination for synchronous resisted rolling; the constants are to be taken from a table like that on page 347. If preferred, Bertin's equation may be used instead of Froude's, especially as it is easier to solve.

As an example choose the case of the *Royal Sovereign*, a sister ship to the *Revenge*. In January, 1893 she encountered a long, low swell, estimated to be 450 to 700 feet long and 6 to 12 feet high. At times she rolled to 16° from the vertical. On another occasion the *Resolution*, a ship of the same class, rolled to 31° from the vertical.

Assuming a time of $7\frac{1}{2}$ seconds for a single roll, or 15 seconds for a double roll, corresponding to the time of a wave from crest to crest, the length of the wave is

$$L = \frac{gT^2}{2\pi} = \frac{32.2 \times \overline{15}^2}{2 \times 3.1416} = 1153 \text{ feet.}$$

If the height of the wave is taken at 0.02 of its length, or 23 feet, the value of the radius of the orbit of a molecule at the surface will be 11.5 feet. The maximum inclination of the wave surface by the equations (4), page 320, and (46), page 261, is

$$\Phi = kr = \frac{2\pi r}{L} = 2\pi \times 0.01 = 0.0628 = 3° \ 36';$$

consequently

$$\frac{\pi}{2}\Phi=\frac{3.1416}{2}\times 3^\circ\,36'=5^\circ\,40'=5^\circ.5 \text{ (nearly)}.$$

The *Revenge* when fully equipped with a displacement of 14,300 tons, without bilge-keels, had a time of rolling of 7.6 seconds, and had an equation of extinction

$$\Delta\Psi=0.0123\,\Psi+0.0025\,\Psi^2$$

in degrees. Consequently

$$0.0025\,\Psi^2+0.0123\,\Psi=5.5,$$
$$\Psi^2+4.92\quad\Psi=2200.$$
$$\therefore\ \Psi=42^\circ.5 \text{ (nearly)}.$$

This is certainly very heavy rolling, heavier than would probably be reached under like circumstances in service, since waves of such length are seldom experienced. With shorter waves the rolling, though heavy, will be less severe and will pass through a series of increasing and decreasing rolls.

After bilge-keels were applied to these ships the equation of extinction became

$$\Delta\Psi=0.650\,\Psi+0.017\,\Psi^2,$$

or, equating to the increment for one roll,

$$0.017\,\Psi^2+0.0650\,\Psi=5.5,$$
$$\Psi^2+3.8\quad\Psi=323.5.$$
$$\therefore\ \Psi=16^\circ \text{ (nearly)}.$$

This shows the great advantage from the use of bilge-keels in the strongest light.

The equation of extinction or the curve of extinction can be applied to finding the maximum rolling among waves of a given type. The method may be conveniently explained for rolling among synchronous waves, for which also numerical calculations can be readily made. For non-synchronous rolling a graphical process due to Wm. Froude will be explained later.

Taking the case of the *Revenge*, it appears that for a wave 1153

feet long and 23 feet high each single roll, corresponding to the passage of half a wave from hollow to crest or crest to hollow, will give an increment of $5^\circ.5$.

Beginning at a hollow with the ship erect and at rest, the first half-wave would give an inclination of $5^\circ.5$ if there were no resistance. The mean roll may be called

$$(0+5.5)\div 2=2^\circ.75,$$

for which the decrement by the equation of extinction is $0^\circ.3$. The ship on the crest of a wave will then have an inclination

$$5.5-0.3=5^\circ.2.$$

If there were no resistance, the next roll would be

$$5.2+5.5=10^\circ.7,$$

the mean being 8° nearly. The decrement corresponding is 1.6, so that the angle at the end of the roll will be $9^\circ.1$. The third roll without resistance wou'd be to the angle

$$9.1+5.5=14^\circ.6.$$

The mean of $14^\circ.6$ and $10^\circ.7$ will be $12^\circ.6$; the corresponding decrement is $3^\circ.5$, so that the angle attained is

$$14.6-3.5=11^\circ.1.$$

The process can be carried on thus step by step until the maximum angle of 16° is attained at which the increment due to half a wave is equal to the decrement for the next roll. The very notable thing is that only three rolls are required to attain an angle of 11°, which corresponds with the fact that ships, especially large and heavy ships, attain large angles of rolling very suddenly when acted on by a long, low swell which synchronizes or nearly synchronizes with the time of rolling. Light ships, which roll quickly, may be steady under the same circumstances.

Froude's Method.—A graphical method of determining the time of rolling of a ship was given by Wm. Froude,* which has the merit

* Proceedings Trans. Inst. Naval Arch., Vol. XVI.

that it permits the introduction of resistance to rolling and can be extended to the rolling of a ship among waves. His method is in fact a general method for the graphical integration of a differential coefficient of the second order, and it is convenient to state the general proposition on which the method depends, and also certain devices for the convenient application of this proposition.

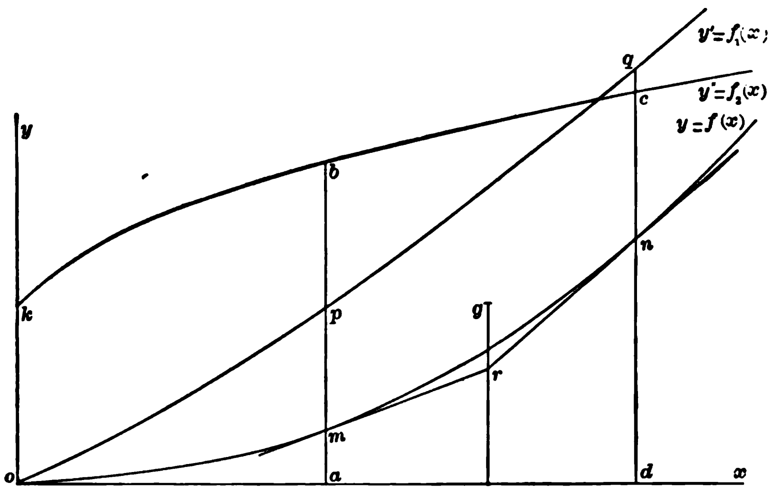


FIG. 155.

In Fig. 155 let bc be a part of a certain curve referred to the axes Ox and Oy; let pq be the integral curve of bc, and let mn be the integral curve of pq. Conversely, pq is the differential curve of mn, and bc is the differential curve of pq. Froude's fundamental proposition is that the tangents at the points m and n meet in a point r on the abscissa of the centre of figure g of the figure $abcd$.

To prove this proposition we may proceed to find the abscissæ of g and r by the usual methods, whereupon it will appear that they are identical. To begin with, it is clear that if the ordinates of the three curves at a given abscissa x are y, y', and y'', then

$$y'=\frac{dy}{dx} \quad \text{and} \quad y''=\frac{dy'}{dx}=\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The area of the figure $abcd$ is

$$A=\int y''dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the moment of that figure about Oy is

$$M = \int xy''dx = \int xdy' = xy' - \int y'dx, \quad . \quad . \quad . \quad (3)$$

the last transformation being obtained by integrating by parts. If the ordinates at the points a and d are indicated by the subscripts 1 and 2, then the area and moment of $abcd$ are

$$A = \int y''dx = y_2' - y_1'$$

and

$$M = x_2y_2' - x_1y_1' - y_2 + y_1,$$

and the abscissa at the centre of figure is

$$x_g = \frac{x_2y_2' - x_1y_1' - y_2 + y_1}{y_2' - y_1'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equations to the tangents at m and n may be written

$$y_1 - y = \left(\frac{dy}{dx}\right)_1 (x_1 - x) = y_1'(x_1 - x) \quad . \quad . \quad . \quad . \quad (5)$$

and

$$y_2 - y = \left(\frac{dy}{dx}\right)_2 (x_2 - x) = y_2'(x_2 - x). \quad . \quad . \quad . \quad . \quad (6)$$

The abscissa of the point of intersection is obtained by eliminating y, giving

$$y_1 - y_2 = x_1y_1' - x_ry_1' - y_2'x_2 + x_ry_2'. \quad . \quad . \quad . \quad . \quad (7)$$

and

$$x_r = \frac{x_2y_2' - x_1y_1' - y_2 + y_1}{y_2' - y_1'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is identical with the expression for x_g.

The following device is convenient for locating the centre of figure of a trapezoid: Through the point c of the trapezoid $abcd$, Fig. 156, draw a horizontal line cp; divide the side bc in halves by the point h, and divide the side ad in thirds at m and n; connect h and m by a line which intersects cp in k; then an ordinate through k will pass

through the centre of figure g of the trapezoid. To prove this construction locate the centre of figure g_1 of the rectangle $apcd$ at its middle, and also the centre of figure g_2 of the triangle pbc on an ordinate g_2m which divides the base cp (or da) into thirds; the centre of figure g of the trapezoid will be at a point which divides g_1g_2 into segments that are inversely proportional to the areas of the rect-

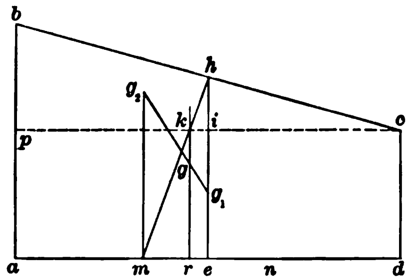


FIG. 156.

angle and the triangle; but these figures having the same bases will have their areas proportional to ei and ih respectively. Evidently the line he is divided in the required proportion at i, and from the construction of the figure the lines hm and g_1g_2 are divided in the same proportion at k and g respectively.

Another device is used by Froude to determine the area of a figure bounded by a curve and two ordinates. Let $abcde$, Fig. 157, be the figure whose area is desired. First join b and d, forming the trapezoid $abde$, and draw the ordinate gf at the middle of its base, and extend it to meet the curve at the point c; divide fc at two-thirds of its height by the point p; then the area of the figure is very nearly equal to $ae \,.\, pg$. This method is correct if the curve bcd is the arc of a parabola whose axis is parallel to cg, for the area of the parabolic segment bcd is two-thirds of the circumscribed

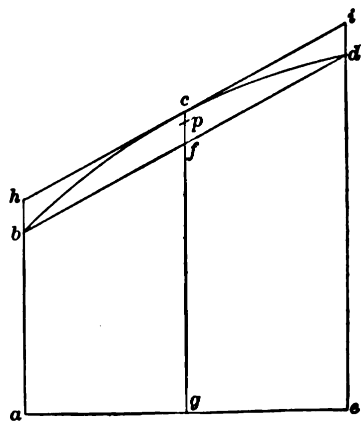


FIG. 157.

parallelogram *bhid*, and consequently the area of the trapezoid and the parobol c segment is equal to the base multiplied by *gj* plus two-thirds of *jc*.

To apply Froude's method to unresisted rolling we may resume the general equation (2) deduced on page 309,

$$\frac{d^2\theta}{dt^2} = -\frac{g(h-a)}{\rho_0^2}\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where θ is the inclination of the ship from the vertical in circular measure, t is the time in seconds, g is the acceleration due to gravity, $h-a$ is the distance of the intersection of the original and new verticals above the centre of gravity (see Fig. 33, page 73), and ρ_0 is the radius of gyration of the hull and its contents in feet. If the weight and location of all the members of the hull and of all its contents are known, the radius of gyration may be calculated directly and introduced into equation (9), though the calculation is laborious and not altogether satisfactory; most commonly this computation is not made, but the radius of gyration is inferred from the time of rolling to very small angles—for example, to one degree—because for so small an angle the rolling is sensibly isochronous and without resistance. The time for a single roll from one side to the other is given by equation (6), page 311,

$$t = \frac{\pi\rho_0}{\sqrt{g(r_0-a)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

under the assumption that the rolling is unresisted and isochronous. Here r_0 is the metacentric radius or r_0-a is the metacentric height, which may be obtained from an inclining experiment. To distinguish this time from the time of rolling to any angle it will be given the subscript zero, and the value of the radius of gyration may be written

$$\rho_0 = \frac{t_0\sqrt{g(r_0-a)}}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Introducing this value into equation (9),

$$\frac{d^2\theta}{dt^2} = -\frac{\pi^2}{t_0^2(r_0-a)}(h-a)\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The righting arms in feet for angles of inclination in degrees are given by the curve of statical stability, which curve will be known for a given ship or may readily be drawn. From that curve we may construct a new curve which will give values of the right-hand member of equation (12) for all values of θ; this curve will be plotted with the inclinations of the ship in circular measure for abscissæ and with the numerical values of the right-hand member of equation (12) to the same scale for ordinates; equations (9) and (12) indicate correctly that the acceleration and the force producing it

FIG. 158.

FIG. 160.

FIG. 159.

are in a contrary direction to the inclination of the ship. Let Fig. 158 give the curve of accelerations for a given ship in terms of the inclination, derived as described from the curve of statical stability; the curve is extended to 0.5, which corresponds to about 28½ degrees.

In Fig. 159 the time of rolling is laid off in seconds on the axis of abscissæ to the same scale as is used for both ordinates and abscissæ in Fig. 158, and the inclinations of the ship in circular measure taken from Fig. 158 are laid off as ordinates, while the corresponding values of the acceleration, $\frac{d^2\psi}{dt^2}$, also taken from Fig. 158, are laid off below the axis OT as is required by their negative sign. It is convenient to begin at the end of a roll where the ship has its maximum inclination Oa because the tangent ab to the curve of inclinations is there horizontal; in the example the maximum inclination is taken as .5 or 28½ degrees, but it may evidently be any desired angle. Take a convenient small interval of time at the end of which the inclination and the acceleration are to be determined; in Fig.

159 this is taken as one second, but in practice it is advisable to take a fraction of a second. The acceleration is laid off at Om for the beginning of this period to correspond with Fig. 158, and the probable acceleration at the end of the interval of time is laid off at $1p$ by guess, and the curve of accelerations is sketched in from m to p. The abscissa of the centre of figure of $Omp1$ is determined, if necessary, by aid of the method of Fig. 156; commonly it may be assumed to be at the middle of $O1$. The mean height of the same figure is to be determined by the method of Fig. 157 if necessary; commonly it may be taken to be equal to un, the ordinate of the curve of accelerations at the middle of the interval of time. By Froude's fundamental proposition the tangents ab and cb at the ordinates Oa and $1c$ meet on the ordinate ub, which is drawn through the centre of figure of $Omn1$; we may therefore draw the tangent ab, which is known to be horizontal, and its intersection b with ub is the point from which the tangent bc is to be drawn. In order to draw the tangent bc we may note that from equations (1), page 357,

$$y' = \frac{dy}{dx} \quad \text{and} \quad y' = \int y'' dx,$$

from which it is evident that the area under the second differential curve is equal to the tangent of the angle which the tangent to the integral curve makes with the axis of abscissæ. Applied to the case in hand, the area of the figure $Omp1$ is the tangent of the angle between the line bc and the axis OT. In this case, since $O1$ is one second, the area of $Omp1$ is numerically equal to its mean height, which may be assumed to be un; we may therefore lay off $On' = un$ and draw $n'1$, and from b draw bc parallel to it; the tangent bc will cut the ordinate $1c$ at a point on the curve. It will be noted that the tangent at c is inclined down towards the axis OT, as it should be to correspond with the negative sign of the value of the acceleration by equation (12). To verify our work so far as we have gone, we may now take in Fig. 158 an abscissa equal to $1c$ and see if the corresponding ordinate is equal to $1p$; should it not be equal it would be necessary to make another estimate of the location of the point p and repeat the construction.

Proceeding, we may estimate the location of the point r after the

second interval and locate the intermediate point q, which in the figure is on a line joining p and r, but which, if necessary, may be located with greater refinement, as already explained. The distance $n'q'$ is now to be laid off equal to vq, and $q'1$ will give the direction of the tangent at d, because Oq' is numerically equal to the sum of the areas $Omp1$ and $1pr2$. We may therefore continue the figure by producing bc to d and drawing de parallel to $q'1$; this latter line can be extended to f on the ordinate at w which is midway between 2 and 3. The construction is to be verified by comparison of $2e$ and $2r$ with Fig. 158, or a new construction is to be made if necessary. The figure in the case in hand is found to close at 3 seconds; that is, the ordinate zero at that point gives ws for the mean height of $2r3$, and when $q's'$ is made equal to ws the line $s'1$ gives such a direction to the tangent drawn from f towards 3 that it crosses the axis at 3. The time of half a roll is, therefore, three seconds, and the time of a single roll is six seconds.

For the sake of simplicity in explanation all the scales of Figs. 158 and 159 are made the same, and as the intervals of time are each one second, the diagrams for tangents are drawn by laying off the mean heights directly at On', $n'q'$, etc. Had the intervals of time been half-seconds, then clearly the areas would have been equal to the mean heights multiplied by one-half the base; that is, the halves of the mean heights would have been laid off in the determination of the directions of the tangents; or if preferred, the mean heights could have been laid off as before and the points like n', q', etc., connected with the point 2. If the intervals are taken as quarter-seconds, or if some other fraction is selected, a corresponding change of construction will be made.

The diagram is likely to be long and attenuated, especially for a slow-rolling ship, in which case it may be contracted by taking the scale for time smaller; this will not change the process of construction in any manner. If it be considered that the diagram Fig. 158 for inclinations and accelerations is inconvenient, then the scale for accelerations $\frac{d^2\phi}{dt^2}$ may be made twice, or several times, as large as the scale for the angles, in which case the base OT in Fig. 159 for laying off angles must be correspondingly increased.

To illustrate all these modifications due to taking fractions of seconds, a contracted scale for time and a magnified scale for accelerations, let it be supposed that the intervals of time are each one-fifth of a second, and that the scale for accelerations is four times the scale for inclinations; then the base for laying off angles will be $5 \times 4 = 20$ seconds laid off from O; as already pointed out, the contraction of scale for time will change the length of Fig. 159, but not its other proportions.

Equation for Resisted Rolling.—Extensive experiments have been made especially by the Froudes and by Bertin on the rolling of ships at sea, and by aid of a provisional equation for resisted rolling a comparison has been made between the theory for rolling among waves and the result of these experiments which shows a fair concordance.

In the first place it may be noted that the rolling will be approximately about an axis through the tranquil point, which must itself be determined experimentally. The moment of inertia of the ship and its contents should be determined with reference to this axis instead of an axis through the centre of gravity, as was done for unresisted rolling. The difference between these moments of inertia and the corresponding radii of gyration is not likely to be large and may commonly be neglected. We may therefore, as on page 360, take for the radius of gyration

$$\rho_0 = \frac{t_0 \sqrt{g(r_0 - a)}}{\pi}$$

where t_0 is the time of rolling for a small angle.

To simplify the discussion let it be assumed that the unresisted rolling of the ship is isochronous, and that the decrease of amplitude for a single roll is small, so that the acceleration may be determined by equation (3), page 311, which may be written

$$\frac{d^2\phi}{dt^2} = -b^2\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$b = \frac{\sqrt{g(r_0 - a)}}{\rho_0}.$$

As required for isochronous rolling, the righting moment at the inclination ψ is

$$D(r_0 - a)\psi; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

D being the displacement of the ship and $r_0 - a$ the metacentric height.

Let it be assumed that the resistance to rolling may be expressed as a function of the first and second powers of the angular velocity, a convenient form being

$$D(r_0 - a)\left\{\alpha\frac{d\psi}{dt} + \beta\left(\frac{d\psi}{dt}\right)^2\right\}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which represents the moment of resistance, where α and β are numerical ratios and where the displacement and metacentric height are introduced for convenience in making comparison with the righting moment.

Finally, let it be assumed that the net work of the righting moment for a single roll (as from port to starboard) is equal to the work required to overcome the resistance during that roll. Now the work required to incline a ship slowly from the erect position to an angle Ψ is the dynamic stability at that angle,

$$D(r_0 - a)\int_0^\Psi \psi d\psi = \tfrac{1}{2}D(r_0 - a)\Psi^2; \quad . \quad . \quad . \quad . \quad (4)$$

and the work of the righting moment during a half-roll from the inclination Ψ may be represented by the same expression. In like manner the work of the righting moment during a half-roll from the erect position to an angle ψ_1 is

$$\tfrac{1}{2}D(r_0 - a)\psi_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Consequently the net work of a roll from the angle Ψ (port) to the angle ψ_1 (starboard) may be written

$$\tfrac{1}{2}D(r_0 - a)(\Psi^2 - \psi_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This net work of the righting moment is to be equated to the work required to overcome the resistance, which may be obtained by integration from equation (3):

$$D(r_0-a)\left\{\alpha\int\frac{d\psi}{dt}d\psi+\beta\int\left(\frac{d\psi}{dt}\right)^2 d\psi\right\}. \quad . \quad . \quad . \quad (7)$$

A convenient form of integral of the equation (1) is

$$\psi=\Psi-\cos bt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

a result that can be verified by differentiating twice and then replacing Ψ by its value from equation (8). From this equation the value of the differential coefficient in equation (7) may be readily derived, being

$$\frac{d\psi}{dt}=b\Psi\sin bt; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

this gives also

$$d\psi=b\Psi\sin bt\,.\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Substituting these values in equation (7), the work to overcome resistance becomes

$$D(r_0-a)\left\{\alpha b^2\Psi^2\int\sin^2 bt\,.\,dt+\beta b^3\Psi^3\int\sin^3 bt\,.\,dt\right\}; \quad . \quad (11)$$

or replacing the square and cube of the trigonometric functions by known values,

$$D(r_0-a)\left\{\tfrac{1}{2}\alpha b^2\Psi^2\int(1-\cos 2bt)dt\right.$$
$$\left.+\beta b^3\Psi^3\int(\tfrac{3}{4}\sin bt\,.\,dt-\tfrac{1}{4}\sin 3bt\,.\,dt)\right\}, \quad . \quad . \quad . \quad (12)$$

so that the general integral becomes

$$D(r_0-a)\left\{\tfrac{1}{2}\alpha b\Psi^2(bt-\tfrac{1}{2}\sin bt)-\beta b^2\Psi^3(\tfrac{3}{4}\cos bt-\tfrac{1}{12}\cos 3bt)\right\}. \quad (13)$$

The inferior limit for t in (13) may be made zero at the beginning of the roll when the inclination is Ψ_0 (port) and the superior limit may be

$$t=\frac{\pi}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

from equation (6), page 311, at the end of the roll when the inclination is ψ_1 (starboard), provided that the time of rolling is for the moment assumed not to be affected by resistance. The introduction of these limits reduces (13) to

$$D(r_0-a)\left\{\frac{\pi}{2}\alpha b\,\Psi^2+\frac{4}{3}\beta b^2\,\Psi^3\right\}. \quad . \quad . \quad . \quad . \qquad (15)$$

Equating the work to overcome resistance to the net work of the righting moment, as given by expression (6), and reducing,

$$\Psi^2-\psi_1{}^2=\pi\alpha b\,\Psi^2+\tfrac{8}{3}\beta b^2\,\Psi^3. \quad . \quad . \quad . \quad . \qquad (16)$$

Let the loss of amplitude for a single roll be represented by

$$\Delta\Psi=\Psi-\psi_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

so that, neglecting the square of $\Delta\Psi$,

$$\psi_1{}^2=\Psi^2-2\,\Psi\Delta\Psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Introducing this value into equation (16) and reducing leads to the equation

$$\Delta\Psi=\frac{\pi}{2}\alpha b\,\Psi+\tfrac{4}{3}\beta b^2\,\Psi^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

A comparison of this equation with Froude's equation for the extinction of rolling on page 346 shows that

$$A=\frac{\pi}{2}b\alpha \quad \text{and} \quad B=\tfrac{4}{3}b^2\beta. \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Replacing b by a value that can be obtained from the application of equation (14) to a very small roll, namely,

$$b=\frac{\pi}{t_0},$$

and solving for the constants α and β,

$$\alpha=\frac{2t_0}{\pi^2}A \quad \text{and} \quad \beta=\frac{3}{4}\frac{t_0^2}{\pi^2}B. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The general equation for unresisted rolling on page 309 may now be made to include resistance to rolling if there be added to the left-hand member the expression (3) for the moment of resistance, affected by a negative sign; thus,

$$D(h-a)\sin\psi=-\rho_0{}^2\frac{D}{g}\frac{d^2\psi}{dt^2}-D(r_0-a)\left\{\alpha\frac{d\psi}{dt}+\beta\left(\frac{d\psi}{dt}\right)^2\right\}. \quad (22)$$

This equation expresses the fact that the righting moment at any instant is equal and contrary to the moment required to overcome resistance and produce acceleration. Replacing α and β by their values in terms of A and B, the constants of Froude's equation of extinction, and using the expression for ρ_0 on page 364, and transforming, gives

$$\frac{d^2\psi}{dt^2}+\frac{2}{t_0}A\frac{d\psi}{dt}+\tfrac{3}{4}B\left(\frac{d\psi}{dt}\right)^2=-\frac{\pi^2}{t_0^2}\cdot\frac{h-a}{r_0-a}\cdot\sin\psi. \quad . \quad . \quad (23)$$

Bearing in mind the numerous approximations and assumptions that have entered into the work of this section, it appears that this equation for resisted rolling, which is proposed by Froude, must be considered to be a provisional equation, the use of which must be justified by comparison with the actual rolling of ships. In his memoir referred to on page 356 Froude gives such a comparison, which appears to justify both his theory of the rolling of ships among waves and his method for the graphical determination of resisted rolling which follows.

In connection with very complete investigations of the rolling of a small wooden vessel, the *Elorn*, MM. Duhil de Bénazé and

Risbec * developed an equation much like (23), but with another term, which is a function of the acceleration.

Extensive theoretical and experimental investigations have been made by M. L. E. Bertin,† Ingénieur de la Marine, on the rolling of ships among waves. His methods differ in many essentials from those of the Froudes, but the general conclusions are much the same.

Froude's Method with Resistance.—Having the equations for resisted rolling in quiet water as given above, Froude's graphical method can be extended to include resistance, which has a relatively small effect and can consequently be treated as a correction. The construction can be carried on as described previously without resistance by aid of diagrams like Figs. 158 and 159, making an estimate of the value of the ordinate $1p$ after the first interval of time as before, and constructing the corresponding ordinate $1c$ for the inclination at the end of that interval; but when the check on the construction is made by the aid of Fig. 158, a correction must be made at the same time for the resistance.

Now in Fig. 159 the quantities On', Oq', etc., are the tangents of the angles which the tangents at c, e, etc., make with the axis of abscissæ, that is, those quantities are the values of the differential coefficients $\frac{d\psi}{dt}$. We may therefore take the values of $\frac{d\psi}{dt}$ from the diagram and compute values of the terms containing $\frac{d\psi}{dt}$ and $\left(\frac{d\psi}{dt}\right)^2$; or we may more conveniently compute values of the sum of the two terms representing resistance for a sufficient number of values of $\frac{d\psi}{dt}$ and plot a diagram like Fig. 160, from which the resistance can be interpolated directly; this diagram is to be drawn with the same scale as is used for the other diagrams, and frequently is added to Fig. 159; the diagram is drawn separately at Fig. 160 for sake of clearness of explanation. For example, the ordinate $1p$ at the end of the first interval of time should be equal to the

* Théorie du Navire, Pollard et Dudebout, Vol. II, page 333.

† N. E. Bertin, Les vagues et le roulis; Données théoretic, etc. Complement à l'étude sur la houle et le roulis.

ordinate from the curve of Fig. 158 at the angle $1c$ *plus* the quantity $n'n''$ for resistance; in like manner the ordinate $2r$ should be equal to the ordinate on Fig. 158 for the angle $1e$ *plus* the quantity $q'q''$, etc.

This explanation is with the primary assumption of the graphical method that Figs. 158 and 159 are drawn with the same scale for both abscissæ and ordinates; if there is any departure from this rule, the terms for resistance must be laid off to the same scale as the acceleration, that is, $n'n''$, $q'q''$, $s's''$, etc., must have the same scale as $1p$, $2r$, etc.

The diagram, Fig. 159, is carried only so far as to make the curves cross the axis of time; they may, of course, be carried beyond the axis, both ψ and $\frac{d^2\psi}{dt^2}$ changing sign. For unresisted rolling the continuations of the curve of inclinations and accelerations will be similar to the portions already drawn, and there is consequently no advantage in continuing the construction. But for resisted rolling both the maximum inclination and the time of rolling are likely to change. For unresisted rolling, and on Fig. 159, it is clear that the acceleration is always zero when the ship is erect; this is not necessarily true for resisted rolling, but will be nearly true in any case and will serve as a guide to the proper signs to attach to the several terms of the general equation for rolling.

Resisted Rolling among Waves.—To extend Froude's graphical rolling among waves it may be noted that, as in the case of unresisted rolling among waves on pages 320, the inclination of the ship is measured from the true vertical, but that the angle θ which determines the righting moment is measured from the normal to the wave surface. The inclination ψ of the ship is, therefore, the sum of the inclination ϕ of the normal to the wave surface and the angle θ which the masts of the ship make with that normal. The value of the inclination of the normal to the wave surface can be computed approximately by the equation (3), page 320,

$$\phi = kr \sin kct, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$k = \frac{2\pi}{L} \quad \text{and} \quad c = \sqrt{\frac{g}{k}} \quad \text{and} \quad r = \tfrac{1}{2}h,$$

if h represents the height of the wave from hollow to crest; consequently

$$\phi=\frac{\pi h}{L}\sin\sqrt{\frac{2\pi g}{L}}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this expression h is the height and L the length of the wave, both in feet, and g is the acceleration due to gravity.

By equation (2) the inclination of the wave may be computed for the times represented by the points 1, 2, 3, etc., of a diagram like 159, and they may be plotted as ordinates to the scale used for plotting the ship inclinations. At any time, the difference of the ordinates to the curve of ship inclinations and to the curves of wave inclinations is the angle θ which is to be carried to a diagram like Fig. 158 in order to determine the acceleration $\frac{d^2\psi}{dt^2}$. If account is taken of resistance, we may construct a diagram like Fig. 160, with ordinates equal to $\frac{d\psi}{dt}$ and abscissæ equal to the sums of the terms representing resistance, for use in correcting the ordinates of acceleration and resistance. Strictly we should consider the motion of the water as well as that of the ship in determining the effect of resistance, but such refinement was not considered necessary by Froude, and would be very difficult, if not practically impossible.

CHAPTER X.

RESISTANCE OF SHIPS.

By the term resistance of a ship is meant the force required to maintain a certain speed of the ship through the water; the force is treated as an external force, as, for example, the pull of a tow-line. When a ship is propelled by some internal motor the force exerted (for example, the thrust of a screw propeller) differs from the resistance just defined on account of certain reactions to be investigated later.

A ship which is held at rest in a stream of water flowing uniformly has the same relative condition with regard to the water as though it were towed through still water at a speed equal to the velocity of the current; it is sometimes convenient to treat the resistance of the ship in this way.

Knot and Mile.— Distances at sea are measured in nautical miles, there being sixty nautical miles to a degree of the equator, and this makes the mile 6080.27 feet. Speeds at sea are stated in nautical miles per hour, but it is common to call the unit a knot instead of a mile; this comes from determining speed by the log-line, which has knots tied at intervals. Sometimes distances, especially distances sailed by a ship, are stated in knots; as when a steamship is given a radius of action of a certain number of knots. Distances and speed on inland waters (lakes and rivers) are given in statute miles of 5280 feet. The nautical mile is consequently nearly one-sixth longer than the statute mile.

Kinds of Resistance.—The resistance of a ship may be separated into four several kinds: (1) stream-line resistance, (2) direct or eddy-making resistance, (3) wave-making resistance, and (4) frictional resistance. The first three kinds of resistance are due in one way

or another to pressure or its effect, and the last kind is due to rubbing or friction.

Stream-lines.—A discussion of stream-lines, of sources and sinks, and of the stream-line function has been given on page 24, and should be read again before going on with the application of the methods of stream-lines to the discussion of the resistance of a ship. For sake of simplicity we shall consider only plane stream-lines, which are described by points moving in a plane. A graphical representation of plane stream-lines can be made by drawing stream-lines at such intervals that the flow through the intermediate elementary stream-lines shall be equal. Where the stream-lines in such a diagram are near together the velocity in the streams is high, and where the stream-lines are wide apart the velocity is low.

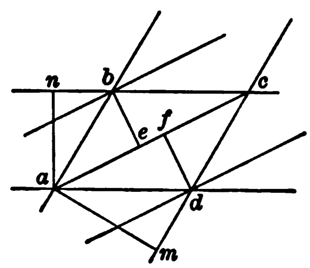


FIG. 161.

Composition of Stream-lines.—Before taking up an analytical discussion of stream-lines it may be interesting to give a very simple graphical composition of them. Let nc and ad represent stream-lines bounding an elementary stream with the velocity of flow v, due to a steady current. Let ab and mc represent stream-lines bounding an elementary stream with the velocity v', produced at the same place by another steady current. If the lengths ab and ad are taken short enough, they may be treated as straight lines even though the paths of a particle are really curved. The velocity of flow in an elementary stream is inversely proportional to the width, consequently

$$v : v' :: am : an.$$

From the similarity of the triangles anb and amd,

$$am : an :: ad : ab.$$

Consequently the sides of a parallelogram made by the crossing of two elementary streams can be taken to represent the velocities in the streams to which they are parallel; that is,

$$v = ad \quad \text{and} \quad v' = ab.$$

The effect of the simultaneous action of two steady currents on a particle of water may be determined by compounding the velocities due to the individual steady currents, using the parallelogram of velocities. Thus the resultant velocity of a particle at a, Fig. 161, is

$$v_c = ac$$

when affected simultaneously by two steady currents having the velocities v and v'.

Now the velocity of an elementary stream is inversely proportional to its width, and conversely the width of a stream is inversely as its velocity. Drawing the perpendiculars df and be to the diagonal ac,

$$df = be;$$

and from the similarity of the triangles anc and bec,

$$be : an :: bc : ac :: v : v_c,$$

from which it is evident that be or fd is the proper width of the composite elementary stream. Therefore the composite system of stream-lines can be constructed by drawing diagonals through all the intersections of the component systems of stream-lines. It can be further shown that there is no flow across the line ac, and t hat it is indeed a stream-line; for the velocity ad may be resolved into the components af and fd, and ab may be resolved into the components ae and eb; the resultant velocity along the diagonal is, therefore,

$$v_c = ae + af = ae + ec = ac,$$

as already determined, but the velocity across ac is

$$be - fd = 0,$$

as it should be.

Stream-lines from a Source.—From the definition of a source on page 48 the stream-lines are straight lines radiating from the source; and in like manner stream-lines converge toward a sink.

Assuming that the depth of the streams is one foot and that the flow from a source is

$$2\pi s, \qquad (1)$$

then s is called the strength of the source. In like manner $-s$ is the strength of a sink. Let a circle be drawn about a source with radius of one foot, then an arc one foot long will correspond to the angle unity. A vertical cylindrical surface drawn to the bottom of the stream through such an arc will have an area of one square foot, and the flow through it will be s cubic feet per second; this gives a concrete meaning to the term strength of a source.

Let the polar coordinates of a point P, with the origin at a source O, in Fig. 162, be r and θ; the flow through a cylindrical surface with radius unity and angle θ will be

$$\psi = s\theta; \quad . \quad . \quad . \quad (2)$$

and if A is any point on the line OX, the flow from the source at O across the curve AP will be represented by equation (2) because both A and P are on stream-lines which pass through the ends of the arc with length θ on the radius unity. Consequently the stream function at P when referred to the point A, due to the source at O, is represented by equation (2). Moreover, this function is not changed in value by changing coordinates, and we may use polar or rectangular coordinates at will and may assume the origin at our convenience.

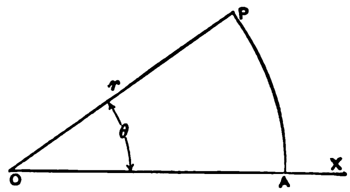


FIG. 162.

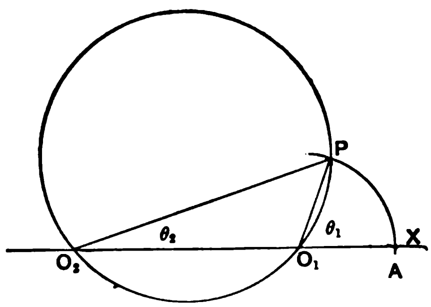


FIG. 163.

Combination of Source and Sink.—In Fig. 163 let O_1 be a source with the strength s, and let O_2 be a sink with the strength $-s$. The stream function at a point P with reference to a point A on the line O_1O_2 due to the source at O_1 will be

$$\psi_1 = s\theta_1,$$

and the stream function at the same point due to the sink at O_2 will be

$$\psi_2 = -s\theta_2,$$

and adding these two functions will give for the function at P under the influence of both the source and the sink

$$\psi = \psi_1 + \psi_2 = s(\theta_1 - \theta_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If ψ is made constant, then

$$\theta_1 - \theta_2 = \psi \div s = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and this condition is fulfilled by a circle passing through O_1, O_2, and P, as shown in Fig. 163; consequently the stream-lines due to the influence of a source and a sink having equal strength are circles through the source and sink.

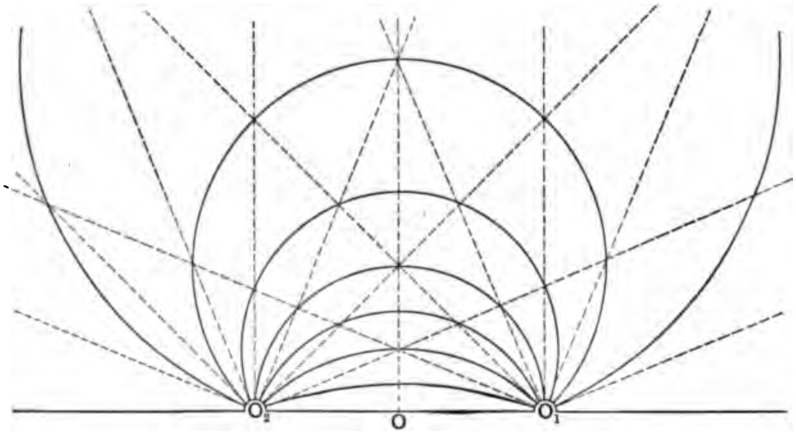


FIG. 164.

Fig. 164 shows a series of circular stream-lines which represent the flow from a source at O_1 to a sink at O_2. The smallest circular stream-line has its centre at O half-way between O_1 and O_2; stream-lines outside of this circle are large arcs of circles like O_1PO_2, Fig. 163; and stream-lines inside of this minimum circle are short arcs like the one lying below the axis of the same figure. Inspection shows that a given circular stream-line passes through the opposite corners of quadrilaterals formed by stream-lines radiating from the source and toward the sink; in fact the circular stream-lines can be so drawn by the graphical method above.

Uniform Flow.—Fig. 165 represents a uniform flow in the direction of the axis OX, the stream-lines being drawn at uniform intervals. If the velocity of the flow is u_0, then the flow across a curve AP, A being on the axis, is

$$\psi_3 = u_0 y, \quad . \quad . \quad . \quad (5)$$

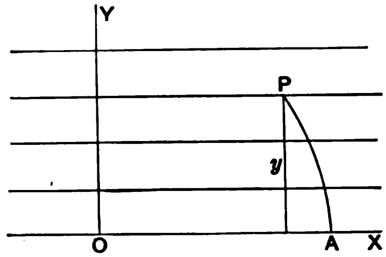


FIG. 165.

it being assumed as in the previous discussion that the stream is one foot deep. If the velocity is negative, the flow is toward the left, and the stream function is negative for positive values of y.

Combination of Source, Sink, and Uniform Flow.—Fig. 166 shows a graphical combination of the flow from a source to a sink with a uniform flow; it is obtained by superposing the lines which represent a uniform flow on the circles (like those in Fig. 164) which represent a flow from a source to a sink and then drawing stream-lines through the resulting quadrilaterals. Both Fig. 164 and Fig. 166 have the stream-lines widely spaced for sake of clearness. For

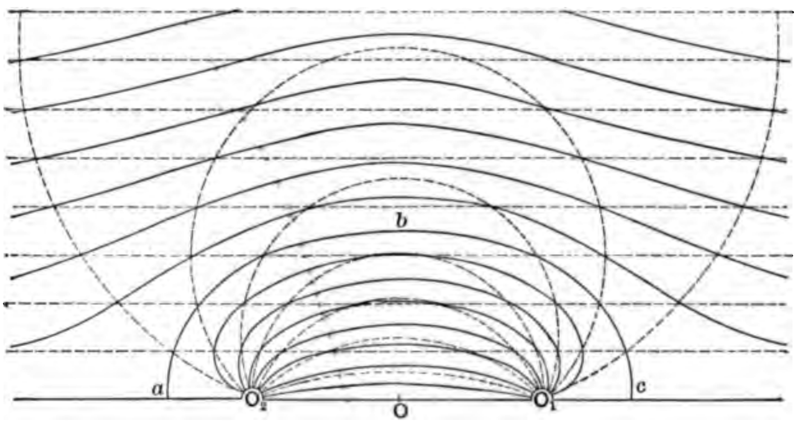


FIG. 166.

a proper construction the spacing should be much nearer. Both figures show only half of the diagrams, the other half below the axis O_1O_2 being symmetrical.

Inspection of Fig. 166 shows that the stream-lines are divided

into two groups by an oval, of which half is drawn. Lines inside this oval are paths of particles which flow from the source to the sink; they do not mingle with particles of the steady flow. Lines outside the oval are paths of particles of the steady flow, that are deflected by the action of the source and sink, but do not cross the oval and mingle with the water that flows from the source to the sink.

Since particles of water flow along stream-lines, but do not cross them, the flow in a stream would not be changed by replacing the ideal surface bounding a stream by frictionless material boundaries. Consequently the boundary of any stream may be considered to be material, and in particular the oval in Fig. 166 may be considered to be a frictionless solid. In such case all the interior stream-lines vanish together with the source and the sink, and the steady stream will flow past the solid and be deflected by it. Streams near the ends of the oval are broadened, but streams near the middle are narrowed, and the average velocity of flow past the solid is greater than u_0, the original velocity of the stream, on account of the obstruction offered by the material body.

From the longitudinal symmetry of the oval it may be inferred that the pressure at the two ends is the same and consequently the resultant pressure is zero, and, as there is no friction, there is no tendency of the stream to move the solid; consequently such a solid placed as shown in a steady stream of frictionless fluid will remain at rest. But the relative motion of the stream with regard to the solid could be obtained by moving the solid with the velocity u_0 toward the right, and it may be inferred that such a solid once set in motion in a frictionless fluid would move with uniform velocity without the application of any force. This is, of course, only a special case of unresisted motion at uniform velocity in a straight line.

The oval of Fig. 166 is not adapted to propulsion at the surface of the water where waves are set up as a consequence of the variation of pressure due to changing velocity as water flows past a ship, but it may suggest proper forms for the bows of submarine boats and automobile torpedoes; the sterns of such vessels must be comparatively long and tapering to avoid eddies. Since the stream-

lines of Fig. 166 are plane lines, while the lines about an immersed body like a submarine boat are stream-lines in space, the form cannot be taken as the most desirable longitudinal section for such a boat. A simple discussion of stream-lines flowing in space from a source to a sink will be given later which has a more direct application to this case.

The stream function at any point of the fluid influenced by the combination of a steady flow with an equal source and sink is obtained by adding the stream functions due to the individual influences. If the strength of the source is s and that of the sink is $-s$, while the velocity of the steady flow is $-u_0$, then from equations (3) and (5) the stream function at a given point is

$$\psi = -u_0 y + s(\theta_1 - \theta_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the origin of coordinates is taken at O, Fig. 167, half-way between the source and the sink (at O_1 and O_2 respectively), and if the half-distances are called $+a$ and $-a$, then

$$\theta_1 = \tan^{-1}\frac{y}{x-a},$$

$$\theta_2 = \tan^{-1}\frac{y}{x+a},$$

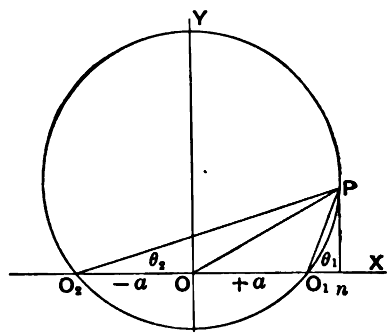


FIG. 167.

and consequently, from equation (6),

$$\psi = -u_0 y + s\left\{\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right\}. \quad . \quad . \quad . \quad (7)$$

Another form of the equation can be obtained by considering that

$$\theta_1 - \theta_2 = O_2PO_1 = O_2Pn - O_1Pn = \tan^{-1}\frac{x+a}{y} - \tan^{-1}\frac{x-a}{y},$$

so that from equation (6)

$$\psi = -u_0 y + s\left\{\tan^{-1}\frac{x+a}{y} - \tan^{-1}\frac{x-a}{y}\right\}. \quad . \quad . \quad . \quad (8)$$

Making the stream function constant in equations (7) and (8) gives the following forms for the equation to a stream-line, either

of which may be used as convenient:

$$-u_0 y + s\left\{\tan^{-1}\frac{y}{x-a} - \tan^{-1}\frac{y}{x+a}\right\} = \text{const.}, \quad . \quad . \quad (9)$$

$$-u_0 y + s\left\{\tan^{-1}\frac{x+a}{y} - \tan^{-1}\frac{x-a}{y}\right\} = \text{const.}; \quad . \quad . \quad (10)$$

but neither equation can be used for tracing the corresponding curve, as the only way of solving one of the equations is by guessing at corresponding values of x and y and then trying to see if they satisfy the equation. Practically the stream-lines are always drawn by the graphical method of Figs. 164 and 166. It may be seen that the equations satisfy some of the known conditions. Thus if y is made very large in equation (10), the inverse tangents approach zero and that equation reverts to equation (5), which represents a uniform flow; that is, at a great distance to one side of the axis the uniform flow is not disturbed by the source and sink. In like manner, if x is made very large in equation (9) it reverts to equation (5), which shows that at a great distance before or after the source and sink the uniform flow is undisturbed.

To get the dimensions of the oval of Fig. 166, Rankine resorts to the following method. In the first place differentiate equation (7) with regard to y, obtaining

$$\frac{\partial\psi}{\partial y} = -u_0 + s\left\{\frac{x-a}{(x-a)^2+y^2} - \frac{x+a}{(x+a)^2+y^2}\right\}. \quad . \quad . \quad . \quad (11)$$

But if z be changed to y in equation (26), page 250, the component velocity parallel to the axis of x becomes

$$u = -\frac{\partial\psi}{\partial y},$$

so that

$$u = u_0 - s\left\{\frac{x-a}{(x-a)^2+y^2} - \frac{x+a}{(x+a)^2+y^2}\right\}. \quad . \quad . \quad . \quad (12)$$

Apply this equation to a particle which approaches the oval of Fig. 166, along the axis of x; this may be done by making y equal to zero, giving

$$u = u_0 - \frac{2as}{x^2-a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For values of x greater than a, u will be numerically less than u_0; that is, the particle slows down as it approaches the oval. The graphical construction of the oval shows that it is at right angles to the axis of x where it crosses it, so that the component velocity of a particle along that axis must be zero at that point. Applying this conclusion to equation (13) by making u equal to zero and solving for the corresponding value of x,

$$x_0 = +\sqrt{a^2 + \frac{2as}{u_0}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

To get a value for the half-breadth of the oval note that the equation to the stream-line of a particle which was originally on the axis of x can be obtained by making the constant equal to zero in equations (9) and (10); this is at once evident in case of equation (9) because a possible solution must be $y=0$. Equation (10) appears to lead to an indeterminate result when y is made zero; but since the real value of the parenthesis must be the same for both equations, the indeterminate expression arising in the latter equation must be equal to zero. If the constant in equation (10) is made zero and x is also made equal to zero,

$$y_0 = \frac{2s}{u_0} \tan^{-1} \frac{a}{y_0},$$

or

$$\tan \frac{u_0 y_0}{2s} = \frac{a}{y_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

an equation that can be solved by trial.

It may be noted that equation (14) for determining the length of the oval contains four quantities: u_0, the velocity of the uniform flow; s, the strength of the source; a, the half-distance between the source and sink; and x_0, the half-length of the oval. If any three are assumed, the fourth may be computed.

Variation of Pressure.—To find the variation of pressure along a stream-line refer to the equation of equilibrium on page 240, namely,

$$\frac{\partial p}{\partial x} = -\frac{w}{g} \frac{d^2x}{dt^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and a similar equation,

$$\frac{\partial p}{\partial y}=-\frac{w}{g}\frac{d^2y}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The complete differential of p is

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy.$$

Replacing the partial differentials by their values from equations (16) and (17) and preparing for integration,

$$\int dp=-\frac{w}{g}\int\frac{dx}{dt}\frac{d^2x}{dt^2}dt-\frac{w}{g}\int\frac{dy}{dt}\frac{d^2y}{dt^2}dt. \quad . \quad . \quad (18)$$

$$\therefore\ p=-\frac{w}{2g}\left(\frac{dx}{dt}\right)^2-\frac{w}{2g}\left(\frac{dy}{dt}\right)^2+c,$$

$$p+\frac{w}{2g}(u^2+v^2)=c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where c is the constant of integration. It will be convenient to represent the resultant velocity of a particle by a single character V so that

$$V^2=u^2+v^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and equation (19) may be written

$$p+\frac{wV^2}{2g}=c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Equation (21) applies to a particle at any point along a stream-line, for example, at the point (x_1, y_1); consequently

$$p+\frac{wV^2}{2g}=p_1+\frac{wV_1^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the discussion of the combination of a uniform flow with a source and sink the reference-point (x_1, y_1) may be taken at a great distance from the source, where $V_1=-u_0$ and $p_1=p_0$; the pressure p_0 being the hydrostatic pressure at the plane under discussion when the uniform flow is not disturbed by the source and sink. For that case equation (22) becomes

$$p+\frac{wV^2}{2g}=p_0+\frac{wu_0^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

To find the maximum and minimum pressures along a stream-line it is sufficient to determine the minimum and maximum values of V for that line. In Rankine's discussion of the combination of a steady flow with a source and a sink, he determines the loci of minimum and maximum velocities by the usual methods. The work is tedious and does not now appear to be profitable. A locus of minimum velocities is found at each end of the oval in Fig. 166, beginning on the axis and extending as a slightly curved line which makes an angle of less than 45° with that part of the axis of x which is external to the oval. Another locus of minimum velocity is found on the axis of y, that is, at the middle of the oval, extending a definite distance from the oval; beyond that distance the axis of y is a locus of maximum velocity. Between the loci of minimum velocities at the end and near the middle of the oval there is a locus of maximum velocity. To interpret these conclusions we may imagine that a vertical cylinder having a horizontal section like the oval of Fig. 166 is fixed in a uniform stream with a velocity u_0 towards the left. The hydrostatic pressure at the surface of the water at a great distance from the oval is, of course, zero. Wherever there is a diminution of velocity there will be a positive hydrostatic pressure, and, on the other hand, wherever the velocity is greater than u_0 there is a negative hydrostatic pressure, or in other words the pressure of the water is less than that of the atmosphere. If the water is free to follow the influence of hydrostatic pressure, it will rise in a wave at each end of the oval, the crest of the swell spreading in a v or u with the arms extending forward at the head of the oval. There will be a less prominent swell at the middle of the oval in its immediate neighborhood, which will die away at a little distance and then change into a hollow. Between the crest near the head and the swell near the middle of the oval there will be a hollow which at a little distance from the oval will coalesce with the hollow at the middle already mentioned. We may readily calculate the height of the crest at the head of the oval because the water there comes to rest, giving for the height in feet

$$h = \frac{u_0^2}{2g}.$$

Rankine's Stream-lines for Ships.—It is clear from inspection that the oval of Fig. 166 is not adapted for the water-lines of a ship. In his investigation of the stream-lines about such an oval, Rankine determined that the velocity of gliding of a particle along certain parts of lines at a little distance from the oval was slow and therefore favorable, as it would give little friction. These lines were, of course, infinite curves beginning and ending at a great distance in the straight lines of uniform flow. Nevertheless Rankine concluded that desirable parts might be selected for water-lines of a ship, and it is likely that his opinion was influenced by the fact that the parts of such lines which he selected resembled the hollow water-lines then advocated by Scott-Russell. It is now well known that hollow water-lines are not desirable, and they appear only where the bow and stern of a ship are faired into a deep stem and stern-post.

Taylor's Stream-lines for Ships.—By a very ingenious graphical method Naval Constructor D. W. Taylor, U.S.N., has succeeded in drawing stream-lines of forms which are adapted to the water-lines of ships. A full description will be found, together with several examples and convenient tables and diagrams, in the XXXVth volume of the "Transactions of the Institution of Naval Architects."

In the first place it is clear that the method of combining a source and a sink can be extended so as to be applied to two sources and two sinks. In fact Rankine made such a combination with the idea of producing a figure more nearly like a ship's water-line than the oval of Fig. 166. The analytical work is necessarily involved, or if a graphical process is chosen, like that used for combining a steady flow with a source and a sink, the work is very tedious. Of course several sources and several sinks can be combined with an ever-increasing complexity. To gain an idea of the action we may suppose each source to be represented by a small hole in the bottom of a shallow tank into which water shall flow under a steady pressure, while the sinks may be represented by holes through which water is withdrawn. Suppose now that instead of several holes for sources we imagine a narrow slit having a varying width; under the same pressure the flow through any small length of the slit will be proportional to its width. In like manner a slit may make a continuous sink through which the water is withdrawn. Mr. Taylor

takes the slits adjacent, so that the sink begins where the source ends. In Fig. 168 let such a combination of a continuous source

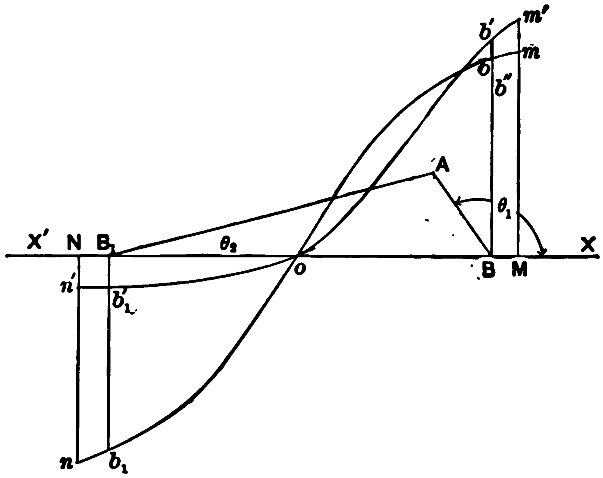


FIG. 168.

and sink extend from M to N on the axis XX'; the strength of the source increases from O to M, and at any point is represented by the corresponding ordinate of the curve Om; in like manner On represents the strength of the sink. The curve nOm is symmetrical in Fig. 168, and that is the simplest arrangement; in any case the area of OMm and ONn must be equal in order that the flow toward the sink shall be equal to the flow from the source. At any point on B the ordinate Bb represents the flow per unit of length of the slit at B divided by 2π, as required by the definition on page 374.

The stream function at the point A per unit of length of the slit at B is

$$Bb\theta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and may be represented by the ordinate Bb'; in like manner a sufficient number of ordinates for points like B along the line MN can be obtained, those between O and N being laid off below the axis as the ordinate for the strength at a given point like B_1 is B_1b_1, drawn downward to represent the flow toward a sink. Construct the curve $n'b_1'Ob'm'$; its ordinate at any point will represent the stream function at A per unit of length of the slit at that point. The area

of the figure $Om'M$ minus the area of $On'N$ represents the resultant stream function at A. Having determined the stream functions for a sufficient number of points like A, it is possible to obtain a series of points by interpolation which shall have the same stream function, and through these points a stream-line can be drawn. Stream-lines found in this manner correspond to the circles of Fig. 164; though curious, they are not useful for the present purpose.

To obtain stream-lines suitable for the waterlines of a ship, the flow from a continuous source to a continuous sink is to be combined with a uniform flow. The stream function for a stream having the uniform velocity $-u_0$ is $-u_0y$, so that if this quantity is added to the stream function for the point A, Fig. 168, previously determined by the graphical method, the sum will be the resultant function at A due to the combination of a uniform flow with the flow from a continuous source to a continuous sink. Having the resultant stream functions at a sufficient number of points, like A, the points at which the function has the same value may be determined by interpolation and a stream-line can be drawn. That stream-line which is drawn through points having zero stream function will lie on a closed curve with its ends on the axis of x. A vertical cylinder with the zero stream-line for its section will be a body which has no resistance to propulsion in a frictionless fluid. Such stream-lines may be made to take any desired form by selecting appropriate curves of strength of source in Fig. 168, and may readily be made to take shapes closely like the water-lines of ships which are known to have high speed without excessive powers. If any conclusion can be drawn from the previous statement, it is that the stream-line resistance of well-formed ships is small, and that conclusion is probably just. It must, however, be borne in mind that ships below the water-line are drawn in toward the keel, and stream-lines about them do not lie in horizontal planes. Moreover, the stream-lines at the surface are affected by the waves that accompany a ship at high speed.

Mr. Taylor has devised certain ingenious methods of abridging the labor of construction and of controlling the dimensions of the zero water-line. He also has methods of determining the velocity at a point from the stream function, and knowing the velocity he

can determine the pressure by aid of equation (23), page 382. The excess of pressure near the bow he assumes to have the effect of producing a bow-way, and the deficit near the middle, of producing a hollow, and he computes the elevation and depression at those points by the usual hydraulic formula.

For example, he finds that a ship 200 feet long and 40 feet beam, will, at a speed of 18 knots, have a bow wave 7 feet high and a depression amidships of 6 feet. The bow wave he considers reasonable, but the depression he thinks is excessive.

Stream-lines about Solids of Revolution.—The discussion of stream-lines about a solid of revolution moving in the direction of the axis of revolution can be reduced to problems in plane geometry, because it is evident that any particle of water which is disturbed by the body will move in a plane passing through the axis of revolution. This conclusion holds only for a single solid moving in undisturbed liquid. As in the case of plane stream-lines, it appears that a convenient treatment can be made by the aid of the combination of a uniform flow with a source and a sink.

Uniform Flow.—If an axis of reference be chosen parallel to the direction of the uniform flow, then a point at a distance y from that axis may have a circle drawn through it by revolving the ordinate y about the given axis. The flow through that circle under the influence of the uniform flow having the velocity u_0 parallel to the axis will be

$$\psi = \pi u_0 y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This function may be attached to the point $(x_1 y)$ and may be treated as is the stream function for plane stream-lines; we may, if we choose, call it the stream function and represent it by ψ, though the function and its conception are usually limited to plane stream-lines.

Source.—If the liquid is supposed to flow from a point radially in all directions in space, then the total flow from that source may be written

$$\psi = 4\pi s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where s is called the strength of the source. A sphere at one foot from the source will have the superficial area 4π; and if, at that surface, the velocity is s feet per second, then the volume flowing

through that sphere will be represented by the expression (2). This discussion is equivalent to that for plane stream-lines in a stream one foot deep.

The assumption of flow from a source in space leads to the same impossibility that was found in the discussion of a source in a plane, namely, that at the source the velocity must be infinite. It is more difficult to state an approximation for a flow from a source in space than for a flow from a source in a plane.

A sink is a point toward which liquid flows, and its strength is considered to be negative.

In Fig. 169 let O be a source and OX an axis drawn through it; if p is a point at one foot from the source, and if the radius Op makes the angle θ with the axis OX, then the flow through the spherical surface generated by revolution of the circular arc pa about the axis of OX will be

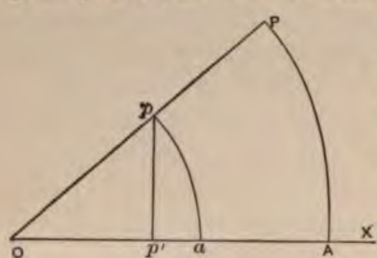


Fig. 169.

$$2\pi s(1-\cos\theta), \quad . \quad . \quad (3)$$

that is to say, it will be equal to spherical area multiplied by s, which is the velocity at a foot from the source. If a point P be chosen anywhere on the radius Op, and another point A be taken on the axis OX, and if any curve AP be drawn connecting these points, then the flow through the surface generated by revolving AP about OX will clearly be the same as that through the spherical surface generated by ap; because, the flow from O being radial, the flow in either case is bounded by a conical surface generated by revolving OP about the axis OX. The flow through the surface generated by AP may be treated as a stream function, and may be represented by ψ and may further be attached to the point P, which has the coordinate x and y or r and θ. It is also clear that the value of the function will not be affected by moving A along the axis OX, provided it does not pass the source; again, the value of the function will be the same for all points along the line OP.

Combination of a Source and a Sink.—Let Fig. 170 represent a source at O_1 and a sink with equal strength at O_2, and let a point

O, midway between O_1 and O_2, be taken for the origin of coordinates in the plane through the source, sink, and the point P, OX being the axis through the source and sink. The stream function at the

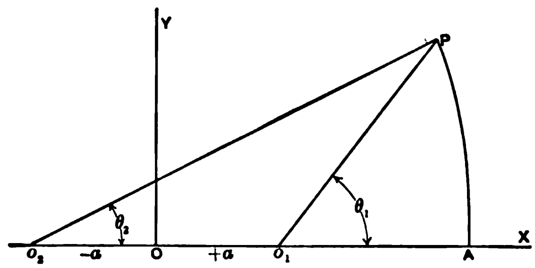


FIG. 170.

point P under the influence of the flow from the source at O_1 will be

$$2\pi s(1-\cos\theta_1),$$

and the stream function at that point from the influence of the sink will be

$$-2\pi s(1-\cos\theta_2).$$

The combined action of the source and sink will be

$$\psi=2\pi s\{(1-\cos\theta_1)-(1-\cos\theta_2)\},$$

or

$$\psi=2\pi s\,(\cos\theta_2-\cos\theta_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A little consideration will show that the same process may be used for any number of sources and sinks located on the axis OX, provided that all are at one side of the curve AP. It is further clear that the stream function is independent of the system of coordinates and is not changed by shifting the coordinates, provided the sources and sinks are unaltered.

It is not difficult to transform equation (4) so as to express it as a function of the coordinates x and y of the point P referred to the origin at O. Thus,

$$\cos\theta_1=\frac{x-a}{\sqrt{(x-a)^2+y^2}}, \quad \cos\theta_2=\frac{x+a}{\sqrt{(x+a)^2+y^2}},$$

and consequently

$$\psi = 2\pi s \left\{ \frac{x+a}{\sqrt{(x+a)^2+y^2}} - \frac{x-a}{\sqrt{(x-a)^2+y^2}} \right\}. \quad . \quad . \quad . \quad . \quad (5)$$

If ψ be made a constant in either equation (4) or equation (5), we get the equation to a stream-line in the plane through the axis OX and the point P. The stream-line for any point in the circle generated by revolving the point P around the axis OX will be like that through P, and will, of course, lie in the plane through the axis and that point. The stream-lines, therefore, form a surface of revolution. Equation (5) is not convenient for use in constructing stream-lines, but it is not difficult to devise a graphical construction based on equation (4).

Suppose that ψ in equation (4) is constant, then

$$\cos \theta_2 - \cos \theta_1 = \frac{\psi}{2\pi s} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If θ_1' is taken to be 90°, then $\cos \theta_1' = 0$, and

$$\cos \theta_2' = \frac{\psi}{2\pi s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This case is illustrated by Fig. 171, where P' is on a vertical through θ. Connect P' with O_2, and then

$$\cos \theta_2' = \frac{2a}{O_2P'} = \frac{\psi}{2\pi s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$O_2P' = \frac{4a\pi s}{\psi} = R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

To proceed with the construction take a point P' on O_1P' and draw a circle through it from the centre at O_2; draw any radius $O_2N = R$ at the angle θ_2; through N draw a vertical MNL, and from O_1 draw a circle with a radius $O_1L = O_2P' = R$; then P is a point on the stream-line through P', for

$$\cos \theta_2 - \cos \theta_1 = \frac{O_2M}{O_2N} - \frac{O_1M}{O_1L} = \frac{2a}{R}, \quad . \quad . \quad . \quad . \quad (10)$$

so that the stream function for P is the same as for that at P'. The same construction can be made for as many points as may be required, and a smooth curve may be drawn. For an angle θ_2'', greater than θ_2', the angle θ_1'' will be greater than $90°$ and $\cos\theta_1''$

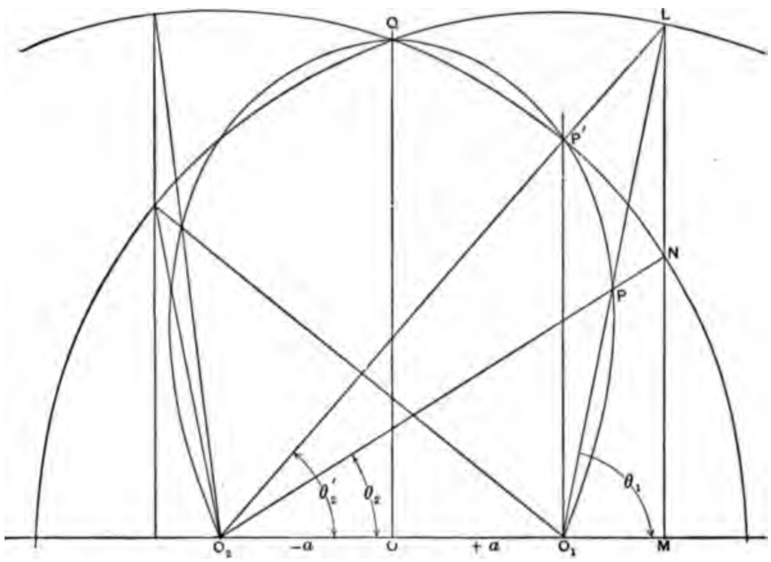


FIG. 171.

will be negative, as it should be, but the construction when made will be sufficiently evident. It will also be evident that Q, the intersection of the two circles, will be a point on the curve, for then

$$\cos\theta_2''' = \frac{a}{R} \quad \text{and} \quad \cos\theta_1''' = \frac{-a}{R},$$

which gives

$$\cos\theta_2'' - \cos\theta_1'' = \frac{2a}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

A series of curves like in Fig. 171 takes the place of the circles of Fig. 164 for the combination of a source and a sink in a plane. In order that the intervals shall represent equal increments of the function ψ, it is sufficient to compute values of the radius R in equation (9) with ψ_1 equal to ψ_1, $2\psi_1$, $3\psi_1$, etc.

Source, Sink, and Uniform Flow.—Let the velocity of the uniform flow be $-u_0$ at a great distance from the source. Then at a given point P the stream function for the uniform flow will be

$$-\pi u_0 y^2.$$

The stream function for the combination of a source, a sink, and a uniform flow will be

$$\psi = 2\pi s(\cos\theta_2 - \cos\theta_1) - \pi u_0 y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This equation can readily be expressed in rectangular coordinates as follows:

$$\psi = 2\pi s\left\{\frac{x+a}{\sqrt{(x+a)^2+y^2}} - \frac{x-a}{\sqrt{(x-a)^2+y^2}}\right\} - \pi u_0 y^2, \quad . \quad (13)$$

and from this equation it is evident that the terms in the parenthesis vanish when either x or y becomes infinite, as should be the case since stream-lines at a great distance from the source are not appreciably affected by the influence of the source and sink.

Equation (13) is not convenient for tracing the stream-lines. Instead we are constrained to use the graphical method suggested by Fig. 166, except that the circles are to be replaced by curves like that in Fig. 171, which should be properly spaced, as explained in connection with that figure. The horizontal lines are to be spaced by making

$$y = \sqrt{\frac{\psi}{\pi u_0}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

as indicated by equation (1) and by giving to ψ the several values ψ, 2ψ, 3ψ, where ψ_1 is to be the same as for the combination of the source and sink. As in Fig. 166, stream-lines are drawn through the quadrilaterals formed by the superposition of the straight lines representing a uniform flow, on the curves representing a flow from a source to a sine. One of the stream-lines will be a closed oval, and this oval, if revolved about the axis OX, will generate a surface of revolution which may be replaced by a frictionless solid. This frictionless solid, if set in motion with a uniform velocity u_0 toward the right, will produce stream-lines in planes through its axis some-

what similar to those shown by Fig. 166. Such a solid would be proper for a submarine boat or an automobile torpedo, if those bodies moved in a frictionless fluid, except that the blunt ends might give rise to excessive pressures.

Extension of Taylor's Method.—A very evident extension of Taylor's method for drawing stream-lines for ships leads to forms that appear to be adapted for submarine boats and torpedoes.

As before, let XX', Fig. 168, be the locus of a continuous source and sink just as in the previous discussion, except that stream-lines are considered to radiate from any given point in all directions instead of being confined to a plane. Assuming as before that the ordinate Bb represents the strength of the source per unit of length, then the stream function at a point A will be

$$2\pi Bb(1-\cos\theta_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Representing this stream function by an ordinate Bb', we may get a curve from which the stream function at A may be obtained by graphical integration. To the stream function thus obtained there is to be added the stream function

$$-2\pi u_0 y^2$$

due the uniform flow. By interpolation, points having the same stream function may be determined and a stream-line in a diametral plane can be drawn. The stream function zero will give a closed curve, which, when revolved about the axis, will generate a surface across which there is no flow. This surface is a figure which will have no resistance in a frictionless fluid, and would appear to be proper for the form of a submarine boat or an automobile torpedo. It has in practice been found that the head of such an automobile torpedo may with advantage be made blunt, thus giving good carrying capacity without too great length; but the tail is usually well tapered to carry the propellers to a suitable distance from the midbody. Even if there were no propellers it would be well to taper the tail in order that the torpedo should be steady on its course and to avoid an eddy astern. The discussion of stream-lines indicates increased pressure astern, and eddies usually occur where there is a reduction of pressure. If eddies

occur about a short submarine boat which is driven at relatively high speed, they are likely to be started amidships and then drift astern. Such an action has been experienced with certain short and full-bodied armored vessels, which in consequence steered badly and especially were unsteady on a course; the remedy was found in lengthening the run. If the strength curve, corresponding to the curve nOm, Fig. 168, is properly chosen, the contour of the section of a submarine boat may be made blunt at the bow and tapering at the tail, it being essential only that the area indicating inflow shall be equal to that indicating outflow.

By taking a proper form of strength curve the section of the surface of revolution can be varied at will and made to approach almost any form that is likely to be chosen for submarine boats which have circular transverse sections. All such bodies will consequently have stream-lines about them, and will not be accompanied by eddies unless they are unduly short or are driven at too high speeds, and all will consequently have little if any stream-line resistance. If well immersed they will have no wave-making resistance, and consequently the resistance will be mainly, if not entirely, frictional. A body which is entirely immersed will have a larger wetted surface than a well-formed floating body of the same displacement and general form, and will be likely to have more resistance.

Stream-lines for Ships.—It does not appear that curves derived from stream-lines nor curves suggested by them are useful in designing ships that navigate the surface of the sea. While it is true that stream-lines can be produced which resemble the water-lines of well-formed ships, it is probable that the shape of the hull will be controlled rather by the effect which the bow will have in making surface waves, and by the necessity of so shaping the stern that the propellers will be well supplied with water. There is, moreover, direct evidence that the stream-lines about a ship are not plane figures; on the contrary, the water at the bow yields horizontally and the water at the stern comes up from below. The direction of stream-lines along the hull has been traced by forcing corrosive fluid from tubes protruding through the skin of the ship, with the result named; that is, the trail of the fluid near the bow is nearly

horizontal, amidships it follows a diagonal, and astern it is nearly vertical.

If any conclusion is proper with regard to stream-lines about a ship, it is that there is little, if any, resistance due to inequality of pressure at the bow and stern of a well-formed ship.

Eddy-making Resistance.—A flat surface like a wide square stern-post gives a resistance like that found to the movement of a flat plane set square across the direction of motion; the resistance for such a plane may be of the first three kinds mentioned on page 372, namely, stream-line resistance, wave-making resistance, and eddy-making resistance. The last, eddy-making resistance, comes from breaking the steady flow of stream-lines into vortices of broken water. Any blunt form like a rounded stem may also produce eddies.

While eddies may be formed near the bow, they are much more likely to be found near the stern in the form of broken water after a wide stern-post, or about the brackets which support the propeller-shafts for twin propellers. Short full ships which are driven at relatively high speeds are likely to have large eddies at the stern which interfere with steering.

It is estimated by Froude that eddy-making resistance does not exceed five per cent of the power required to propel a ship, for well formed ships; it is probably much less for fine steel ships. It is customary to neglect the resistance due to eddy-making or to count it in with frictional resistance, which it most resembles.

Wave-making Resistance.—Any ship that is propelled at speed is accompanied by a system of waves that move along with the ship, and are maintained by the power which propels the ship. Just how these waves are formed and what their exact character is, cannot be stated with certainty. There is some evidence that the bow of the ship starts a wave of the character of a solitary wave; if so, the wave is quickly changed into an undulating wave, as will appear in the further description of the system of waves. On the other hand, a system of stream-lines drawn around a figure, like a water-line of a ship, shows that there is a reduction of velocity near the bow, an increase amidships, and a decrease near the stern. The decrease near the bow and stern is accompanied by an increase

of pressure, for an elementary stream, like water flowing through a pipe, will show an increase of pressure when the transverse section is increased and the velocity is decreased; conversely, the increase of velocity amidship is accompanied by a reduction of pressure. This action is shown in an exaggerated manner by Fig. 166, where the wide spacing of stream-lines ahead of the oval shows a large decrease of velocity and increase of pressure, while the crowding of water-lines at the middle shows an increase of velocity and a corresponding decrease of pressure. Now water at the surface of the sea is affected by the uniform pressure of the atmosphere, which tends to keep it horizontal; if the pressure near the bow of a ship is greater than the normal pressure, water will rise up and form a wave, and a decrease of pressure will form a hollow. Such action is considered to be sufficient to account for the formation of waves.

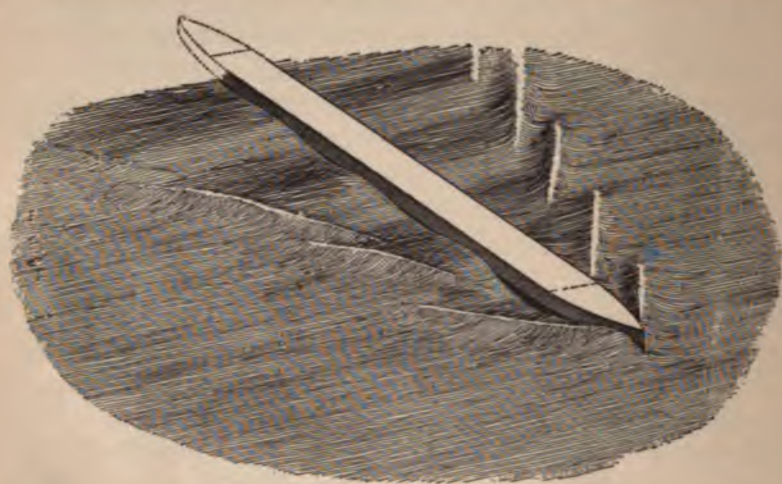


FIG. 172.

The form of the system of waves moving with a ship is shown by Fig. 172, taken from an article by William Froude.* The drawing was made from observations of a model with a very long middle body, which so separated the bow and stern waves that the former were well developed before there was any interference with the

* Proc. Inst. Nav. Archts., Vol. XVII.

latter. The bow system is shown alone; the stern system, if developed alone, would resemble it closely, but usually it is so superposed on the bow waves as to confuse its characteristics.

The figure shows two systems of waves, transverse waves and diagonal waves which terminate the transverse waves. These may be more properly considered to be different manifestations of the same influence rather than the effect of different influences. In fact the hydrodynamical investigation of a similar problem, namely, of an oblique line of pressure advancing over quiet water,* gives a wave pattern having a close resemblance to Fig. 172. The first transverse wave is likely to be confused with the accompanying diagonal wave, and in fact the whole bow wave may be ragged and torn, especially for a vessel at high speed; its crest is not found at the stem, but a little distance aft, approaching a quarter of a wave-length. The first well-formed transverse crest is something more than a wave-length from the stem, approaching one and a quarter lengths. This and the succeeding crests are higher near the ship and diminish till they are terminated by the diagonal crests. The length from crest to crest of the transverse waves is the length proper for an oscillating wave which has the speed of the ship. The heights of súcceeding crests diminish as the width measured between the terminal diagonal crests increases, which appears natural because the energy of the wave is spread over a wider area. The diagonal waves are individually sharper and higher at the inner ends and spread out near the outer ends; they are curved concave outward, conforming in this respect to the pattern determined mathematically, which has already been referred to. These diagonal crests diminish in height as they spread out, much as do the transverse waves which they terminate. The successive crests are in echelon, so that the angle which a line drawn through the system makes with the direction of the ship is about half of the angle which an individual crest makes with that direction; but as the individual crest is curved, this statement must be interpreted in a general manner.

It is very difficult, if not impossible, to see correctly the form of the system of waves moving with a ship, either from the deck of the

* Hydrodynamics, Lamb, p. 402.

ship itself, or from a neighboring ship, as the effect of perspective is to distort the appearance. Moreover, the system is seldom well developed near the ship, and appears only at some distance in its wake. The sharp and threatening crests at the forward ends of the diagonal waves are the most notable feature when the system sweeps by a small boat; the succession of several of these crests gives the impression of a succession of parallel waves of decreasing height. The transverse waves may often be seen to advantage from the stern of a ship, but from that position the diagonal waves cannot be well observed. A system of waves made by a model towed in a towing-tank, or otherwise, can be well observed and measurements can be made.

Without attempting to explain or account for this system of waves, there are certain conceptions that may be obtained by aid of the theory of waves which are interesting and may be instructive. In the first place, the theory shows that the group velocity of a system of waves advancing into quiet water is half that of the individual waves; this shows at once why a system of waves is left in the wake of a ship. If confined between limiting side walls, the waves in the wake would maintain their height; but as they are not so confined, they lose height and are dissipated. The discussion of group velocity gives also an idea why the diagonal waves are in echelon. Again, since a group of waves advances with half the velocity of the individual waves, its rate of propagating the wave disturbance can be considered to be half that required to produce a like disturbance at the velocity of the ship with which the waves move. This consideration is equivalent to saying that the ship must yield half the energy (potential and dynamic) of its train of waves. If the energy of the system could be computed, this consideration would lead at once to a method of determining wave-making resistance. Such a method is not now possible, but the idea leads to a method of explaining the effect of wave interference on resistance which is instructive, in that it indicates how to avoid excessive resistance and gives a provisional form to the function for wave-making resistance.

If it be considered that the bow wave when it is first formed can be assimilated to a solitary wave, it is possible to work out a

method of determining the angle which the crest of that wave makes with the direction of the ship. The method,* though not long nor difficult, does not seem to have sufficient importance to warrant its introduction here. Applied to tests on the torpedo-boat ***Bombe*** it gave about 21° for that angle instead of 20° as determined from towing a model. On the other hand, experiments by the Froudes do not show that such a relation exists for large ships in general.

Function for Wave-making Resistance.—The energy stored in a wave length of a trochoidal wave for each foot of width along the crest is, by equation (70), page 265,

$$E = \tfrac{1}{2} w L r_0^2 \left(1 - \frac{2\pi^2 r_0^2}{L^2}\right),$$

where L is the length of the wave from crest to crest; r_0 is the radius of the orbit at the surface, equal to half the height; and w is the density of the water.

Neglecting the second term in the parenthesis, since r is small compared with L, it appears that the energy is proportional to the square of the height and to the length. It is customary to assume that the energy expended by a ship in wave-making, for each wave length traversed, is proportional to the stored energy in that length of the wave. If the wave-making resistance is R_w, then in running the length L the energy expended is

$$R_w L;$$

making this proportional to the energy of the wave,

$$R_w L \propto L r_0^2,$$

or

$$R_w \propto h^2,$$

where h is the height of the wave from hollow to crest.

Now from equations 43 and 44, page 260, the length of a wave is proportional to the square of the speed of the system. If we assume that the height of a wave made by a ship is proportional to its length, then the wave-making resistance can be made pro-

* Théorie du Navire, Pollard et Dudebout, Vol. III, p. 333.

portional to the fourth power of the speed of the ship, and we may make

$$R_w = bKV^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R_w is the resistance in pounds, V is the speed in knots, K is a factor depending on the size and form of the ship, and b is a factor which should be constant if our theory were strictly logical, but which really varies with the form of the ship.

To make the wave-making resistance conform to the theory of mechanical similitude, the factor K, as will appear later, should vary as a linear dimension of the ship. Naval Constructor Taylor * proposes the following form:

$$K = \frac{D^{\frac{2}{3}}}{L}, \quad \text{or} \quad R_w = \frac{bD^{\frac{2}{3}}V^4}{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where D is the displacement of the ship in tons and L is the length in feet. The value of b may vary from 0.35 to 0.45, having a mean value of 0.4. Long fine ships like Atlantic liners may have $b = 0.35$; moderately fine high-speed vessels may have $b = 0.4$; ships broad in proportion to length, especially if fined at the ends, may have $b = 0.45$; freight ships with block coefficients greater than 0.6 may have b larger than 0.45.

Mr. Taylor elsewhere proposes for fast cruisers like the *Columbia*, *New York*, and *Olympia* the form

$$R = 12.5\frac{CDV^2}{L^2},$$

where D and L have the same significance as in the previous equation and C is the block coefficient.

Wave Interference.—Although it is not now possible to determine theoretically the power which must be expended by a ship to maintain the systems of waves which accompany it at high speed, there are certain phenomena of wave interference which can be shown to have a relation to the resistance of a ship which is driven at high speeds. Large ships are seldom driven at such high speeds

* Resistance of Ships and Screw Propulsion, p. 136. Trans. Soc. Nav. Arch. and Marine Engrs., vol. 2, p. 143.

as would develop marked wave interference, and consequently our knowledge of the phenomena is obtained principally from trials of torpedo-boats and from tests of models in towing-tanks. A provisional form of the function exhibiting the effects of wave interference can be obtained; the constants that are required to make it practically useful are not yet determined. It will appear later that the power required to propel torpedo-boats and other fast vessels can be best determined from experiments on models and on boats already constructed by the aid of the mechanical theory of similitude explained on page 410.

The first well-formed transverse wave has its crest something more than a wave length from the stem of the ship, and in like manner the first transverse crest of the stern wave is something more than a wave-length behind the stern-post. The distance from the first bow-wave crest to the first stern-wave crest is somewhat greater than the length of the ship between perpendiculars; this distance, which is called the wave-making length of the ship, varies from 1.05 to 1.1 of the length of the ship, and increases with the speed of the ship. This wave-making length is determined from investigations of the phenomena of wave interference, and is affected to some degree by the obscurity of the general subject, so that it cannot be considered to be definitely determined. At the present time all high-speed ships, though very fine, have slightly full lines at the bow, and the statements regarding the positions of wave-crests refer to ships of that type. Ships with hollow bow lines have the first diagonal crest and all the features of the bow-wave system farther aft than the location indicated, and their wave-making lengths are correspondingly reduced; such a ship appears to gain nothing by the prolongation of the bow which is due to the use of hollow lines; on the contrary it leads only to larger wetted surface and greater frictional resistance.

The bow-wave system spreads out and is wider than the stern system where they interfere, so that the interference is restricted to the parts that overlap; the outer ends of the transverse bow waves beyond the diagonal terminaters of the stern waves are consequently not affected by the interference. If the transverse waves are considered to be trochoidal, so that the energy can be computed

by aid of equation (67), page 266, then if the average height of the first well-formed transverse crest of the bow wave is h_1, and if the length between the diagonal terminaters is l, the energy in that wave is approximately

$$E_1 = \tfrac{1}{8} wLlh_1^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where L is the wave-length corresponding to the speed of the ship, and w is the weight of a cubic foot of water. If the first crest of the stern system has also the length l and the height h_2, the energy of that wave is

$$E_2 = \tfrac{1}{8} wLlh_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

At the place where the first stern wave would appear were there no interference, the bow-wave system will have spread out and the mean height will be correspondingly reduced so that the height may be represented by kh_1, where k is an arbitrary factor less than unity. That part of the bow wave which takes part in interference will have the energy

$$E_1' = \tfrac{1}{8} wLlk^2h_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The uncombined overlapping ends will have the energy

$$E_1 - E_1' = \tfrac{1}{8} wLl(h_1^2 - k^2h_1^2), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

if it be assumed that the real height of the wave is everywhere equal to the mean height.

By equation (5), page 285, the height of the resultant wave after the interference of the first stern-wave crest and the coinciding part of the bow-wave system will be

$$h = \left(k^2h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{2\pi}{n}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (5)$$

where $\frac{1}{n}$ is the difference of phase of the two component waves. The energy of the resultant wave is

$$E_3 = \tfrac{1}{8} wLl\left(k^2h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{2\pi}{n}\right). \quad . \quad . \quad . \quad (6)$$

Adding the energy of the overlapping ends gives for the total energy

$$E=\tfrac{1}{8}wLl\left(h_1^2+h_2^2+2kh_1h_2\cos\frac{2\pi}{n}\right). \quad . \quad . \quad . \quad . \quad (7)$$

It may be assumed that the wave-making resistance is proportional to the quantity in the parenthesis, and that it will vary with the difference of phase $\frac{1}{n}$. If the difference of phase is one-quarter, the wave-making resistance is proportional to

$$h_1^2+h_2^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

that is, the wave-making resistance is then the sum of the resistance due to the bow and stern systems individually. This is what would be found from a model like that shown by Fig. 172 with a very long middle body which would allow the bow-wave system to be dissipated before it interfere with the stern systems. If the difference of phase is unity or any multiple of unity, so that the bow-wave crests coincide with the stern-wave crests, the resistance is a maximum proportional to

$$h_1^2+h_2^2+2kh_1h_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

On the contrary, if the difference of phase is one-half, or any integral plus one-half, so that the bow-wave crests coincide with the stern-wave hollows, the resistance is a minimum proportional to

$$h_1^2+h_2^2-2kh_1h_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

A ship when at slow speed will have several crests of the bow-wave system along its side, and by the time that this system combines with the stern wave its height will be so reduced that the effect of combining with the stern wave will be inappreciable; in fact the entire wave-making resistance will be of little consequence. It may be interesting to note that in such case the resistance will conform to the first condition.

As the speed of the ship increases the first well-formed transverse crest of the bow wave will approach the stern, and the wave-making resistance will become very important. Large increments of power are required for comparatively small increments of speed. The speed corresponding to the worst case of interference with

crests coinciding may be approached by ships, but is seldom if ever realized, except for torpedo-boats and high-speed yachts and launches. The table of wave lengths and speeds on page 261 will give an idea of the limiting speeds for various lengths of ships; the length of the ship may be taken as $\frac{1}{1.05}$ or $\frac{1}{1.1}$ of the length of the wave. Atlantic liners 600 feet long have speeds of 20 to 22 knots per hour. Fast cruisers, 350 to 400 feet long may have equal speed for short runs, but cannot maintain them for long periods.

If by application of sufficient power to a properly formed boat it is driven at a greater speed than that which gives the worst case of interference, we may approach the third condition represented by equation (10) when the hollow of the bow wave coincides with the crest of the stern wave. Additional increments of speed call for large additions of power, but not at such a rate as would be expected from watching the power and speed before the second condition is reached. A torpedo-boat 175 feet long reaches the speed of worst interference at about 18 knots, but it may be driven at 25 or 27 knots. A torpedo-boat destroyer 245 feet long has the maximum interference at about 23 knots, but it may be driven at 30 or 32 knots per hour.

The general conception of wave interference is well sustained by experiments on models, as will appear later.

Frictional Resistance.—The water in contact with the surface or skin of a ship is dragged along and set in motion, a considerable layer of water being affected in this manner, which can be seen near the stern of a long ship as a broken eddying layer more or less mixed with bubbles of air near the surface. Behind the ship the water thus affected forms a wake which may have from 10 to 15 per cent of the velocity of the ship, unless it is affected by the action of a screw propeller.

Our knowledge of the nature and effect of frictional resistance is due to experiments made by Wm. Froude* in a towing-tank and to similar experiments made by Tideman.†

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* Report to the Admiralty, 1872.

† Memorial van de Marine, also Resistance of Ships and Screw-propulsion, Taylor; p. 224.

The following table gives the results of experiments by Froude with various surfaces.

FROUDE'S EXPERIMENTS ON SURFACE FRICTION.

Nature of Surface.	Length of Surface or Distance from Cutwater.											
	2 feet.			8 feet.			20 feet.			50 feet.		
	A	B	C	A	B	C	A	B	C	A	B	C
Varnish......	2.00	0.41	0.390	1.85	0.325	0.264	1.85	0.278	0.340	1.83	0.250	0.226
Paraffin......	1.95	0.38	0.370	1.94	0.314	0.260	1.93	0.271	0.237			
Tinfoil.......	2.16	0.30	0.295	1.99	0.278	0.263	1.90	0.262	0.244	1.83	0.246	0.232
Calico.......	1.93	0.87	0.725	1.92	0.626	0.504	1.89	0.531	0.447	1.87	0.474	0.423
Fine sand....	2.00	0.81	0.690	2.00	0.583	0.450	2.00	0.480	0.384	2.06	0.405	0.337
Medium sand	2.00	0.90	9.730	2.00	0.625	0.488	2.00	0.534	0.465	2.00	0.488	0.456
Coarse sand..	2.00	1.10	0.830	2.00	0.714	0.530	2.00	0.588	0.490			

In the above, for each length stated in the heading—

Column A gives the power of the speed according to which the resistance varies.

Column B gives the mean resistance in pounds per square foot of the whole surface for a speed of 600 feet per minute.

Column C gives the resistance in pounds, at the same speed, of a square foot at a distance abaft the cutwater stated in the heading.

The following table by Tideman gives the friction of ships in sea-water; for ships in fresh water the frictional resistance can be assumed to be proportional to the density.

TIDEMAN'S SURFACE-FRICTION CONSTANTS FOR SHIPS IN SALT WATER OF 1.026 DENSITY

Length of Ship in Feet.	Iron Bottom Clean and Well Painted.		Copper or Zinc Sheathed.			
			Sheathing Smooth and in Good Condition.		Sheathing Rough and in Bad Condition.	
	f	*n*	*f*	*n*	*f*	*n*
10	0.01124	1.8530	0.10000	1.9175	0.01400	1.8700
20	0.01075	1.8490	0.00990	1.9000	0.01350	1.8610
30	0.01018	1.8440	0.00903	1.8650	0.01310	1.8530
40	0.00998	1.8397	0.00978	1.8400	0.01275	1.8470
50	0.00991	1.8357	0.00976	1.8300	0.01250	1.8430
100	0.00970	1.8200	0.00966	1.8270	0.01200	1.8430
150	0.00957	1.8290	0.00953	1.8270	0.01183	1.8430
200	0.00944	1.8290	0.00943	1.8270	0.01170	1.8430
250	0.00933	1.8290	0.00936	1.8270	0.01160	1.8430
300	0.00923	1.8290	0.00930	1.8270	0.01152	1.8430
350	0.00916	1.8290	0.00927	1.8270	0.01145	1.8430
400	0.00910	1.8290	0.00926	1.8270	0.01140	1.8430
450	0.00906	1.8290	0.00926	1.8270	0.01137	1.8430
500	0.00904	1.8290	0.00926	1.8270	0.01136	1.8430

The tables below by R. E. Froude give the friction of paraffin models and of smoothly painted ships.

FROUDE'S SURFACE-FRICTION CONSTANTS FOR PARAFFIN MODELS IN FRESH WATER.

Length of Model in Feet.	Coefficient of Friction. f	Power according to which Friction Varies. n	Length of Model in Feet.	Coefficient of Friction. f	Power according to which Friction Varies. n
2.0	0. 1176	1.94	12.0	0.00908	1.94
3.0	0.01123	"	12.5	0.00901	"
4.0	0.01083	"	13.0	0.00895	"
5.0	0.01050	"	13.5	0.00889	"
6.0	0.01022	"	14.0	0.00883	"
7.0	0.00997	"	14.5	0.00887	"
8.0	0.00973	"	15.0	0.00873	"
9.0	0.00953	"	16.0	0.00864	"
10.0	0.00937	"	17.0	0.00855	"
10.5	0.00928	"	18.0	0.00847	"
11.0	0.00920	"	19.0	0.00840	"
11.5	0.00914	"	20.0	0.00834	"

FROUDE'S SURFACE FRICTION CONSTANTS FOR WELL-PAINTED SHIPS IN SEA-WATER.

Length of Vessel or Model in Feet.	Coefficient of Friction. f	Power According to which Friction Varies. n	Length of Vessel or Model in Feet.	Coefficient of Friction. f	Power according to which Friction Varies. n
8	0.01197	1.825	80	0.00933	1.825
9	0.01177	"	90	0.00928	"
10	0.01161	"	100	0.00923	"
12	0.01131	"	120	0.00916	"
15	0.01106	"	140	0.00911	"
16	0.01086	"	160	0.00907	"
18	0.01069	"	180	0.00904	"
20	0.01055	"	200	0.00902	"
25	0.01029	"	250	0.00897	"
30	0.01010	"	300	0.00892	"
35	0.00993	"	350	0.00889	"
40	0 00981	"	400	0.00886	"
45	0.00971	"	450	0.00883	"
50	0.00963	"	500	0.00880	"
60	0.00950	"	550	0.00877	"
70	0.00940	"	600	0.00874	"

From his experiments on surface friction Wm. Froude deduced the following formula for frictional resistance of ships:

$$R_f = fSV^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where f is the coefficient of friction to be taken from one of the tables on the preceding pages, S is the wetted area in square feet, and V is the speed of the ship in knots per hour.

Froude tacitly assumes that the velocity of water past the wetted surface is the same as the velocity of the ship, which is probably as good as any other convention for the purpose of computing resistance, though it is clear that the average velocity of the water following the form of the ship must be greater. The chief criticism of the method is the extension of experiments on comparatively small surfaces and at moderate speeds to ships 600 to 700 feet long with speeds of 20 knots per hour and over. Additional experiments on a large scale are very much to be desired.

Wetted Surface.—In order to calculate the frictional resistance of the ship it is necessary to know the surface of the under-water body or the wetted surface. Since the surface of a ship cannot be properly developed, a close determination of the actual surface involves considerable trouble; various approximate methods are in common use which will be explained below.

Approximate Development.—Take a diagonal at about the turn of the bilge, making it as nearly perpendicular as possible to the contour of the sections, and draw and fair as in the usual manner; an ordinary bilge diagonal is usually sufficient. Develop this diagonal as a straight line; this will, of course, give increasing intervals between stations near the ends. At each station on the body plan measure the half-girth, from the diagonal to the water-line, and from the diagonal to the keel, and lay off these distances, the one above and the other below the developed diagonal. Fair curves through the ends of the ordinates will inclose a space which is very nearly one-half the true wetted surface.

Sometimes, when a rough approximation is desired, the half-girths are measured and laid off on a base-line equal to the length of the ship. The area thus obtained is too small, the error being nearly in proportion to the difference between the lengths of a bilge diagonal and length of the ship.

Taylor's Method.—The most complete method for determining the wetted surfaces of a ship is that proposed by Naval Constructor

Taylor.* Let Fig. 173 represent the forebody plan of a ship drawn as usual to the outside of the frames; properly the body plan should be drawn to the mean of the outside plating, but the gain in accuracy is too slight to pay for the trouble of drawing a new body plan. The half-girths (such as ED) are divided into the same number of equal parts; three in the figure, but five or more are desirable in practice. Lines like FG and HI are drawn through the divisions for convenience in locating them. At a point, as C, a normal line is drawn to the contour ECD terminated by the adjacent half-breadths at A and B; then a plane passed through AB parallel to the plane of the paper will cut the skin of the ship nearly in a straight line. Fig. 174 shows the section revolved into the plane

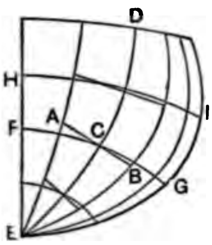

FIG. 173.

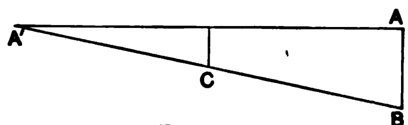

FIG. 174.

of the paper, $A'B$ being the section of the skin of the ship in its true length. Now make a scale with twice the distance between stations for unity and divide it decimally. AB (Fig. 174) measured by such a scale gives the tangent of the angle $AA'B$, and the secant of the angle gives the ratio

$$A'B:AA'.$$

The upper table on page opposite gives the tangents advancing by hundredths and the corresponding secants.

In practice the scale can be set by the eye normal to the contour of the station without drawing the line AB; the work is most expeditiously done by two computers, one reading tangents on the body plan and the other taking secants from the table.

In the same way secants are taken at each division of a half-girth and the mean secant is obtained by adding together half the secants at the ends and all the secants at intermediate divisions, and

* Trans. Soc. Nav. Archts. and Marine Engs., vol. I.

then dividing by the number of intervals. The half-girths are multiplied each by its mean secant and the modified half-girths are then laid off from a base-line at their proper stations. A curve joining the ends of the modified half-girths will give a figure representing the wetted surface.

NATURAL TANGENTS AND SECANTS FOR WETTED-SURFACE CALCULATIONS.

Tangent.	Secant.	Tangent.	Secant.	Tangent.	Secant.	Tangent.	Secant.
0.010	1.000	0.160	1.013	0 310	1 047	0.460	1.101
0.020	1.000	0.170	1.014	0.320	1.050	0.470	1.105
0.030	1.000	0.180	1.016	0.330	1.053	0.480	1.109
0.040	1.001	0.190	1.018	0.340	1.056	0.490	1.114
0.050	1.001	0.200	1.020	0.350	1.060	0.500	1.118
0.060	1.002	0.210	1.022	0.360	1.063	0.510	1.123
0.070	1.002	0 220	1.024	0.370	1.066	0.520	1.127
0.080	1.003	0.230	1.026	0.380	1.070	0.530	1.132
0.090	1.004	0.040	1.028	0.390	1.073	0.540	1.137
0.100	1.005	0.250	1.031	0.400	1.077	0.550	1.141
0.110	1.006	0.260	1.033	0.410	1.081	0.560	1.146
0.120	1.007	0.270	1.036	0.420	1.085	0.570	1.151
0.130	1.008	0.280	1.038	0.430	1.089	0.580	1.156
0.140	1.010	0.290	1.041	0.440	1.093	0.590	1.161
0.150	1.011	0.300	1.044	0.450	1.097	0.600	1.166

Approximate Equations.—In the design of a ship it is important to estimate the wetted surface before the lines of the ship are drawn; for this purpose a number of empirical equations have been devised. Let L be the length, B the beam, and H the mean draught of a ship, all in feet; let D be the displacement in tons, and K the block coefficient; then we use one of the following equations:

Taylor: Surface $= C\sqrt{DL}$.
Normand: Surface $= 1.52LH + (.374 + 0.85K^2)LB$.
Mumford: Surface $= L(1.7H + KB)$.

The constant C in Taylor's equation is to be taken from the following table:

$B \div H$	C	$B \div H$	C	$B \div H$	C
2.0	15.63	2.5	15.50	3.0	15.62
2.1	15.58	2.6	15.51	3.1	15.66
2.2	15.54	2.7	15.53	3.2	15.71
2.3	15.51	2.8	15.55	3.3	15.77
2.4	15.50	2.9	15.58	3.4	15.83

An elaborate investigation of these rules and some others by Profs. Durand and McDermott * showed that they gave the following errors when applied to several types of vessels.

ERRORS OF APPROXIMATE EQUATIONS FOR WETTED SURFACE.

Type.	Length	Beam.	Draught	Displacement.	Block Coefficient.	Wetted Surface.	Error Per Cent. Taylor.	Normand.	Mumford.
Ocean liner....	520.0	70.0	27.0	14596	0.520	43298	−1.32	+0.06	−0.95
Ocean liner....	458.0	56.5	26.0	12122	0.631	36586	+0.17	−0.14	0.00
Paddle steamer	240.0	27.9	8.0	897	0.586	7191	+2.52	+2.57	−0.04
Steam-yacht...	195.0	25.0	13.6	1014	0.537	7145	−2.69	−1.55	−0.52
Cruiser.	313.8	48.0	22.7	5294	0.541	20065	+0.10	+0.76	−0.50
Freighter......	267.5	36.0	18.0	3477	0.703	14955	+0.33	0.00	0.00
Great Lake freighter	298.7	40.9	15.9	4577	0.825	18590	−2.50	1.40	−12.55

Mechanical Similitude.—In the comparison of one ship with another, or of results obtained from experiments on a model with the performance of a ship, we make use of the theory of mechanical similitude, which enables us to determine from the relations of the fundamental units, such as length, mass, and time, what should be the proper ratios of properties like volume, velocity, and power. The fundamental units will be assumed to be the foot, the mass of one pound, and the second. All properties will be made to depend on accepted combinations of these units. The table on page opposite gives the ratios for a number of properties.

The form of the functions showing the relation of volumes and superficial (or other) areas of similar bodies (like a ship and her model) are obvious from the idea of geometrical similarity. Velocity is derived at once from the definition that it is the space passed over in a unit of time; and in like manner acceleration comes from the definition which makes it the gain in velocity per unit of time. Angles are supposed to be measured by the ratio of arc to radius, which gives a semi-circumference equal to 180°, and unity corresponds with about 57°. Angular velocity is, of course, inversely proportional to the time, and angular acceleration is derived by the differentiation of angular velocity with regard to time. Since force is measured by the acceleration imparted to a mass, it is properly

* Trans. Soc. Nav. Archts. and Marine Engs., vol. 2.

TABLE FOR MECHANICAL SIMILITUDE.

Properties.	Symbol.	Function.
Linear dimension, length	l	
Time	t	
Mass	m	
Surface or area	A	l^2
Volume	V	l^3
Angle	α	
Velocity	v	$\frac{l}{t}$
Angular velocity	ω	$\frac{1}{t}$
Acceleration	a	$\frac{l}{t^2}$
Angular acceleration	$\frac{d\omega}{dt}$	$\frac{1}{t^2}$
Force or weight	f or w	$\frac{ml}{t^2}$
Work	W	$\frac{ml^2}{t^2}$
Power	P	$\frac{ml^2}{t^3}$
Density	d	$\frac{m}{l^1l^2}$
Pressure per square foot	p	$\frac{m}{lt^2}$
Moment of inertia	I	ml^2
Momentum		$\frac{ml}{t}$
Moment (of a couple)		$\frac{ml^2}{t^2}$

expressed by the product of the mass by the acceleration; replacing the latter by $\frac{l}{t^2}$ gives the function in the table. Work is obtained by the multiplication of force by distance, and power by the division of work by the time in which it is done. Other properties are obtained in like manner.

If other than fundamental units are desired in the function, they may be introduced by proper substitutions. For example, force may be expressed as a function of the velocity by replacing $\frac{l}{t}$ by v, giving

$$f \propto \frac{ml}{t^2} = \frac{ml^2}{lt^2} = \frac{mv^2}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Equations of Condition.—In applying the theory of mechanical similitude to special cases, there are certain conditions peculiar to each case which may be expressed by equations of condition.

At any given place the acceleration due to gravity is constant, which gives a constant in place of $\frac{l}{t^2}$ in the function for weight. So that

$$w \propto m;$$

that is, the weight is in proportion to the mass. If the acceleration is represented by g, we get the usual form

$$w = mg. \qquad (2)$$

In dealing with floating bodies on the same fluid the density is constant. With a constant for the acceleration due to gravity we have

$$d \propto \frac{m}{l^2 t^2} = \frac{ml}{l^3 t^2} = \frac{mg}{l^3},$$

$$d \propto \frac{m}{l^3};$$

but as the density is constant, $m \div l^3$ is constant; therefore also

$$m \propto l^3, \qquad (3)$$

and consequently

$$w \propto l^3; \qquad (4)$$

or, expressing the weight of the ship as its displacement in tons,

$$D \propto l^3. \qquad (5)$$

All this is merely expressing formally the fact that these known properties of floating bodies are provided for in the theory of mechanical similitude.

Let us now see what will be the consequence of imposing the condition that resistance shall be proportional to the displacement. This may be expressed as

$$R \propto D \propto l^3,$$

while expression (1) makes a force vary as $\frac{mv^2}{l}$; if both conditions are fulfilled, then

$$l^3 \propto \frac{mv^2}{l},$$

whence

$$ll^3 \propto mv^2;$$

but the condition that density shall be constant makes m proportional to l^3, so that

$$v^2 \propto l$$

$$v \propto \sqrt{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Expressing this as a proportion,

$$v : v_1 : : \sqrt{l} : \sqrt{l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If two similar ships have the lengths l and l_1, then speeds proportional to the square roots of their lengths are called proportional speeds.

From direct experiments the form for surface or frictional resistance has been found to be

$$R_f = fSV^n,$$

where n is something between 1.8 and 2. This form is incompatible with the theory of mechanical similitude, and the assumption that the resistance shall vary as the displacement. The latter makes

$$R_f \propto l^3,$$

and the latter makes

$$v^2 \propto l,$$

and both combined make

$$R_f \propto l^2 l \propto l^2 v^2 \propto SV^2,$$

since the surface is proportional to the square of a linear dimension. Consequently the proper form for the surface resistance appears to be

$$R_f = fSV^2,$$

with f a constant coefficient of friction; but not only is n less than 2, but both f and n decrease with the length of the ship.

Turning now to the form assigned to the wave-making resistance,

$$R_w = b\frac{D^{\frac{2}{3}}}{l}V^4;$$

and replacing D and V by functions of l,

$$R_w \propto \frac{l^2}{l}l^2 = l^3;$$

so that the wave-making resistance would conform to the theory of mechanical similitude if b were strictly a constant. But b varies not only with the form of the ship, but also with the speed for a given ship, consequently the wave-making resistance also shows discordance. Nevertheless it has been found that for corresponding speeds the wave-making resistance does conform very nearly to the theory of mechanical similitude.

The assertion that the resistance of a ship is proportional to its displacement, provided that the speed is proportional to the square root of its length is known as Froude's law of comparison; being named for Wm. Froude, who developed it from consideration of stream-line motion. It was considered by him to apply to the wave-making resistance only, which was called the residual resistance since it was obtained by subtracting the surface resistance from the total tow-line resistance. It is sometimes applied to the total resistance for making first approximations.

Extended Law of Comparison.—The form of the function for power in the theory of mechanical similitude is

$$P \propto \frac{ml^2}{t^3};$$

if m is proportional to the cube of the length, then power has the form

$$P \propto \frac{l^5}{t^3}.$$

Applying the condition that the velocity shall be proportional to the square root of the length,

$$P \propto l^2 \frac{l^3}{t^3} = l^2 v^3 \propto l^2 l^{\frac{3}{2}} = l^{\frac{7}{2}}.$$

Further, applying the condition that the displacement is proportional to the cube of the length,

$$P \propto l^{\frac{7}{2}} \propto (D^{\frac{1}{3}})^{\frac{7}{2}},$$
$$P \propto D^{\frac{7}{6}};$$

that is, at corresponding speeds the powers are as the $\frac{7}{6}$ powers of the displacements.

This should apply properly only to the net power applied for propulsion, but as the efficiency of similar engines is nearly the same, it can be and is applied to the indicated horse-power of ships in making first estimates. In estimating the power of a ship from that required for a smaller ship this method always gives excess of power because the surface resistance does not follow the law of comparison. The coefficient of friction for the larger and longer ship is somewhat less, and frictional resistance increases less rapidly than the square of the speed.

Towing-tank.—Experiments on the resistance of models are conveniently made in towing-tanks. The first tank was established by Wm. Froude for the English admiralty; since then tanks have been established for several European governments and for the United States. Such tanks are long enough to give a free run of about 300 feet to the model; to this length are added a basin and a pit at one end for convenience of adjusting the model and overhauling the towing-carriage. The tanks are about 30 feet wide and 10 feet deep. The models are commonly about 12 feet long, without regard to the actual length of the ship represented, and of course are made to scale. The American tank at Washington is arranged for models 20 feet long. The towing-carriage is a bridge which spans the tank just above the water. The carriage for the English tank is made of wood to give combined lightness and stiffness; it is towed by a wire rope wound up by a steam-engine at one end of

the tank. The engine is started when the car is put in motion, and has an ingenious governor, invented by Wm. Froude, which quickly gives a uniform speed to the carriage The carriage of the American tank is a riveted steel girder, which with the towing mechanism is very massive; it is set in motion by electric motors which have abundant power; a double system of powerful brakes brings the carriage gradually and quietly to rest from a high speed. The highest speed of the carriage is required for tests on models of torpedo-boats. If we assign a speed of 30 knots an hour to a torpedo-boat 175 feet long, then a model 20 feet long should have a speed of

$$30\sqrt{20 \div 175} = 10+$$

knots per hour, or 17 feet a second, at which rate it will take about 19 seconds to run 300 feet. The effective time for making a test is something like 10 seconds. For larger ships and for slower speeds the effective time will be several times as long; but in any case automatic recording apparatus for measuring pull on the model and the speed of the carriage are necessary.

In addition to the tests of the force and power required to tow models, these tanks have been used for studying screw propellers and other methods of propulsion. Since the power required to propel ships and their models does not conform to the law of comparison or the theory of mechanical similarity, direct experiments on propulsion of models are not made. The propeller is mounted behind the model and is driven at its appropriate speed; the power required to drive it, its thrust available for propulsion, and its influence on the propulsion of the model, are determined separately, and from these data the performance of the propeller for the ship is inferred. This matter will be discussed later in connection with propulsion. Model tanks have been found especially useful in studying the action of paddle-wheel steamers.

The model is commonly cut from paraffine which is cast on to a basket-work frame. A profiling-machine is used for this purpose which has a pair of rotating cutters that cut down to the contour of the model along a water-line. These cutters are controlled by a pantograph which has the tracing-point carried along water-lines of

the tracing of the ship. By a proper setting of the pantograph the desired size of model can be made from water-lines drawn to any scale. The paraffine between the water-lines is cut away by hand to properly form the model.

Since the climate at Washington is too warm for the use of paraffine models, they are made of wood. A body plan is drawn to the proper scale for the model, from the lines of the ship, by the aid of a pantograph. From this body plan moulds are made which are planked to the true surface of the model, and this serves as a frame from which a profiling-machine forms the model in solid wood; the model is, of course, made of layers to allow the selection of wood of the proper quality. After the model is shaped by the profiling-machine and finished by hand, it is carefully measured on a coordinate frame; since small changes of shape and displacement can be readily allowed for by any method of interpolation, it is more important to know the model exactly than to have it conform exactly to the ship. Of course a model which deviates too much can be corrected or replaced by another.

To get reliable information from model experiments, coefficients of reduction must be obtained by comparing tests on the models with progressive or other speed trials of the ships themselves. Tank experiments that are not corrected by full-size tests are liable to be misleading.

The water in towing-tanks is usually fresh; consequently a correction must be made for density when model experiments are made for sea-going ships. According to the theory of mechanical similitude force and power are proportional to the mass, which in turn is proportional to the density; consequently the resistance and the power to overcome it are proportional to the density of the medium. With equal cubical displacements or equal volumes, the resistance of a model in fresh water and in salt water are proportional to the densities; the resistance in salt water is consequently

$$64.05 \div 62.425 = 1.026$$

as great as in fresh water; this amounts to an increase of about $\frac{1}{40}$.

Froude's Method.—The method of determining the resistance

of a ship from model experiments as developed by Froude is to tow tbe model at various speeds, the maximum being somewhat greater than the expected speed of the ship. The speeds may then be used for abscissæ and the resistances for ordinates of a curve of resistance. It is convenient to take the real speeds of the ship in knots and the corresponding speeds of the model, so that the same scale can be used for abscissæ for both ship and model. Fig. 174 gives the sum of resistance for a model $\frac{1}{16}$ the size of the ship, AA being the curve of resistance of the model in pounds.

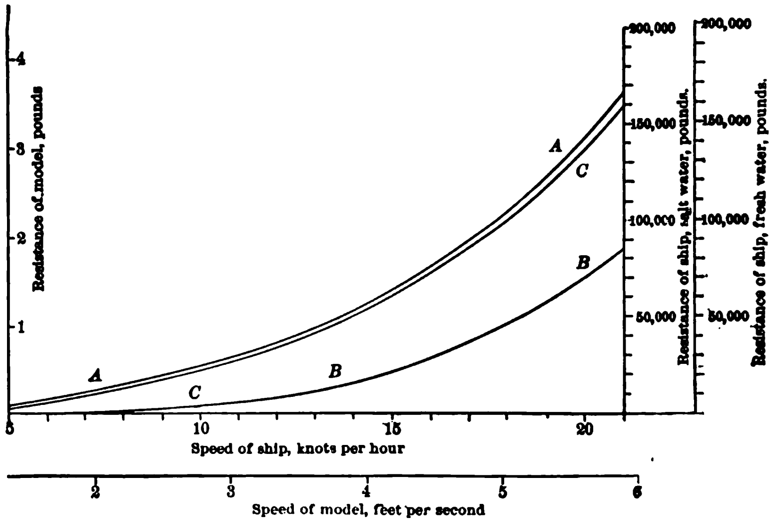

FIG. 174.

The surface resistance of the model in fresh water is now to be calculated by the formula

$$R_s = fS_m V_m^n,$$

where S_m and V_m are the surfaces and velocity of the model. These are subtracted from the total resistances, giving the residual resistances shown by the curve BB. The scale of speed for the ship is

$$\sqrt{\tfrac{1}{16}} = \tfrac{1}{4}$$

the scale for the model, so that the intervals are figured four times as large.

The residual or wave-making resistances of the model are now to be multiplied by the ratio of the displacements of the ship and the model, to get the wave-making resistances of the ship. The factor is here

$$16^3 = 4096.$$

The same result is attained by taking a scale for resistances of the ship $\frac{1}{4096}$ of the scale for the model. The surface resistance of the ship is now to be calculated at the various speeds by the equation

$$R_s = fSV^n$$

and added to the wave-making resistances, giving the curve *CC* for the total resistance of the ship. If the resistance of the ship in salt water is desired, the resistance in fresh water is multiplied by the ratio of the densities, 1.026, as before deduced. This can be done by taking a scale 1.026 times as great for the ship in salt as in fresh water.

Evidently the same method can be applied for estimating the resistance of a given ship from the known resistance of a similar ship. This amounts to analyzing the resistance of a ship into surface resistance and wave-making resistance, and using the latter as a basis for calculating the wave-making resistance of another ship. The method may be applied to a ship of somewhat different form, provided it be through the determination of the factor *b* of the wave-making resistance.

Resistance in Shallow Water.—The resistance of a ship may be notably greater in shallow than in deep water, due to the interference of the bottom with the system of waves which move with the ship. A ship passing over a shoal is likely to slow down appreciably and the engines to show fewer revolutions per minute.

In order that speed trials of a ship shall be satisfactory, it is requisite that the depth of water on the course shall be sufficient. From the discussion of the relative speeds of waves in deep water and in shallow water on page 282 it appears that if the depth of water is equal to half the length of a trochoidal wave, then the speed of wave in such water differs inappreciably from that of the trochoidal wave. It is considered that half a wave length is sufficient depth for speed trials; but the depth should always be large compared with the draught of the ship. The following table gives the depth requisite for various speeds on this basis:

Speed of Ship, Knots.	Minimum Depth for No Change in Resistance, Feet.
10	28
12	40
14	55
16	71
18	90
20	111
22	135
24	160
26	188
28	218
30	250

Solitary Wave Resistance.—In very shallow water a ship appears to form a solitary wave, and as the speed of the ship approaches that of the solitary wave for the depth of the water, given by equation (16), page 304,

$$C = \sqrt{2gD},$$

the resistance increases very rapidly. When the speed of the ship reaches that of the solitary wave the law of resistances changes suddenly, and there is only a gradual increase with speed. At a sufficiently high speed the resistance may be equal to or less than that in deep water.

This action is shown by certain tests made by Captain Rasmussen * on a Danish torpedo-boat. The following table shows the indicated horse-power for the boat at various speeds in water 20 fathoms deep and in 2½ fathoms. The principal dimensions were: length 145 feet 6 inches, beam 15 feet 6 inches, displacement 132 tons.

Speed, Knots.	Horse-power.	
	20 Fathoms.	2½ Fathoms.
10.0	100	100
12.0	160	260
14.0	280	1080
16.0	520	1180
18.0	840	1220
20.0	1320	1380
22.0	1800	1660
23.6	2200	
24.1	....	2200

* Engineering, vol. lviii, 1894.

At the speed corresponding to that of the solitary wave the boat mounts on top of the wave, and at higher speeds it runs away from it and there then appears to be no proper wave formation moving with the boat. The system of waves moving with the boat becomes very unstable at and near the speed of the solitary wave; putting the helm over, for example, changes the wave formation and very much reduces the speed.

A large ship will always show loss of speed in shallow water, as it can never attain the speed of the solitary wave in such a depth of water, even though in deep water it may run at a higher speed. Thus the solitary wave in 5 fathoms or 30 feet of water has a speed of 18 knots. A large ship in deep water may well attain a speed of 18 knots or more, but it cannot in 5 fathoms.

Air Resistance.—The resistance of the air on the out-of-water part of the ship is appreciable, but not large. From experiments by Wm. Froude* it may be expressed by the formula

$$R_a = kAV^2,$$

where A is the transverse projection of the portion of the hull out of water, and V is the velocity in knots, while k is a constant having the value 0.0048. From tests on the *Greyhound* Froude found that at 10 knots the air resistance on the hull was 1.5 per cent of the water resistance; the resistance of the rigging he found about equal to that of the hull. The *Greyhound* had a length of $172\frac{1}{2}$ feet, a beam of 33 feet, and a displacement of about 1100 tons. Modern steamships are 8 or 10 times as long as broad, if not more, and have little if any rigging; their air resistance in a calm is less than one per cent. In a gale the resistance of the wind may be very appreciable. It is not customary to make a separate estimate of air resistance, which consequently goes into the residual resistance along with the wave-making resistance.

* Proc. Inst. Nav. Archts., vol. xv.

CHAPTER XI.

PROPULSION OF SHIPS.

THERE are three ways in which a ship can be made to move through the water: it may be towed, it may be moved by sails, or it may be propelled by some engine or motor in the ship. The propulsion of ships by sails will be discussed in Chapter XIII.

Towing.—The frictional and wave-making resistances of a ship have been sufficiently discussed in the last chapter, and from that discussion the power required for towing can be determined. Except for moving ships in and out of harbor, towing is applied only to barges for lake and coastwise traffic; to a limited extent towing on rapid rivers is carried on by aid of a rope or chain lying in the bed of the stream.

Transportation in barges is used mainly for such crude freight as coal and iron ore. The barges may be old hulks of sailing- or steam-ships, or may be built for the purpose of wood or steel. A large towing-vessel, which may carry a cargo of its own, may take a string of two or three barges. The strength of hull of a barge need not be so much as for sailing- or steam-ships, as they do not have stresses due to wind-pressure on sails or from the action of an engine. There is also considerable convenience in loading or unloading. The speed is usually five to seven knots an hour—considerably less than that of slow freighters.

As will be seen in the discussion of steam propulsion, only 0.50 to 0.60 of the power of the engine is applied to propulsion of the ship. There appears to be a direct gain from towing on this account; but since the power is developed by the engine of the towboat, it is probable that there is only a transfer of a loss from one place to an-

other, since the propeller of a towboat works under disadvantageous conditions.

Towing in Canals.—A vessel which is towed in a restricted channel like a river or canal meets with much greater resistance than it would in open water, on account of the proximity of the bottom and sides. On account of the necessity of passing through locks, the ends of the boat are blunt and the sides are straight; but since the speed is always slow it is probable that the form has a secondary influence on resistance. The following formula is proposed by Elnathan Sweet * from tests on the Erie Canal:

$$R = \frac{.10303sv^2}{r - .597},$$

where s = wetted surface, v = velocity in feet per minute, r = ratio of section of boat to canal. On the other hand some tests on boats in the Seine and French canals appear to show that the resistance is the same for boats having the same transverse section without regard to the length.

Chain Towing.—A system of towing is employed on certain rapid European rivers by aid of a chain lying in the channel of the stream. The towboat picks up the chain, passes it over guide-wheels, and acts upon it by driving-wheels, and finally deposits it again in the channel. One or two barges may be towed behind. The resistance in this case is affected by the narrowness and shallowness of the channel, and also by the rapid current, which not only increases the frictional resistance, but demands additional power to mount the slope of the stream.

Internal Propulsion.—A ship may be propelled by an engine in one of the three following methods: by hydraulic propulsion, by paddle-wheels, and by screw propellers. Thornycroft uses for boats navigating shallow water a form of turbine propeller which appears to be intermediate between a screw propeller and hydraulic propulsion.

A motor or an engine in a vessel can propel it only by acting on the water on which it floats, and must impress a sternward velocity on the water affected. The propelling agent, whether a paddle-

* Trans. Am. Soc. Civ. Eng., vol. ix.

wheel, screw propeller, or the jet of a hydraulic propeller, exerts an effort on the water affected which must be equal and contrary to the effective resistance of the ship. This effective resistance is, in general, larger than the tow-line resistance, since the proximity of the propeller to the ship increases the resistance.

Considering that an effort or force is measured by the momentum imparted, it appears that we may choose whether we will impart a high velocity to a small body of water, or a low velocity to a large body of water. But since the energy imparted to the water is proportional to the square of the velocity, it is apparent that it is desirable to act on as large a body as may be convenient.

Efficiency of Propulsion.—There are two methods of considering the efficiency of propulsion; one considers the mass of water affected by and set in motion by the propelling agent, and the other considers the action of the individual part, i.e., the paddle of a paddle-wheel or the blade of a screw propeller on the water. If we can determine the volume or mass of the water acted upon and the velocity imparted to it, we can at once find the energy expended upon the water, and a comparison with the energy developed by the engine gives the efficiency of propulsion. This method overlooks the energy wasted in friction, and unnecessary disturbance by the propelling agent; moreover, there is but one method of propulsion for which the amount of water acted upon can be estimated with sufficient certainty, that is, for hydraulic propulsion.

The second method has been employed to develop a very complete theory of the screw propeller which takes cognizance of the friction of the water on the blades as well as the propulsive effort. If there were sufficient experimental information, this second method might be applied also to paddle-wheels; at present an inadequate treatment by the first method is usually given.

Following the first method of considering the water acted upon, we may represent the weight of water acted on per second by W, the acceleration of that water by s, and the effort (equal to the resistance of the ship) by R. Then, since a force is measured by the momentum imparted,

$$R=\frac{Ws}{g}. \qquad \qquad (1)$$

If the velocity of the ship is V feet per second, then the work done on the ship in one second is

$$RV = \frac{WsV}{g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

on the other hand the kinetic energy imparted to the water is

$$\frac{Ws^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that the efficiency of propulsion is

$$e = \frac{\frac{1}{g}WsV}{\frac{1}{g}(WsV + \frac{1}{2}Ws^2)} = \frac{V}{V + \frac{1}{2}s}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is apparent that the efficiency increases as s, the velocity imparted to the water, decreases, that is, as the mass acted upon increases.

This efficiency can be called the fluid efficiency, since it takes account only of the energy usefully employed for propulsion and the energy imparted to the water; the latter is commonly said to be wasted. The fluid efficiency must be notably reduced to allow for friction of the engine and for waste of energy in the propelling agent.

Hydraulic Propulsion.—This name is applied to the propulsion of a boat by a stream or streams of water which are directed aft through nozzles. The water is usually taken in through a large scoop-shaped inlet in the bottom of the boat near the middle of its length. The inlet is shaped so that as the boat proceeds its motion may make water flow in. The water taken in is acted upon by a centrifugal pump with a vertical or slightly inclined axis, and is delivered through two nozzles near the water-line. These nozzles may be formed in the structure of the boat and are then in the shape of channels leading aft near the midship section. Other exits directed forward

provide for backing the ship. Large straightway valves control the exits, which afford a ready means of manœuvring the boat without stopping or changing the direction of rotation of the engine. Some boats have been made with a nozzle in the form of an elbow projecting beyond the side of the boat; this nozzle can be turned so as to be directed aft or forward, and thus the boat can be manœuvred.

A certain hydraulically propelled boat made by Thornycroft was investigated by Sydney Barnaby,* who found that the fluid efficiency of the jet was, by equation (4), 0.71, that the efficiency of the centrifugal pump (including resistances of friction in passages) was 0.46, and that the mechanical efficiency of the engine was 0.77. The combined efficiency was

$$0.77 \times 0.46 \times 0.71 = 0.254.$$

This may be compared with a screw propeller of a boat of the same size and form for which Mr. Barnaby gives: efficiency of propeller 0.65, efficiency of engine 0.77, combined efficiency

$$0.77 \times 0.65 = 0.5.$$

It appears likely that the efficiency of the engine was estimated from the combined efficiency, which was assumed to be 0.5 for the screw-propelled boat; for the mechanical efficiency of the engine is likely to be 0.80 to 0.85. Again, the efficiency of the engine for hydraulic propulsion was taken to be the same as that of the screw engine. The velocity of the jet of the hydraulic propulsion was determined from its pressure on a small plane placed across its direction.

The poor effect of hydraulic propulsion is to be attributed in large degree to the centrifugal pump, which cannot have a good efficiency. Had a turbine pump of 0.80 efficiency been used, the combined efficiency would have been

$$0.77 \times 0.80 \times 0.71 = 0.44.$$

* Marine Propellers, p. 171.

The hydraulically propelled boat made a speed of 12.6 knots per hour with 167 indicated horse-power, while the screw boat made 17.3 knots with 170 horse-power. It is, however, fair to note that the hydraulic boat had a greater beam and had a displacement of 14.4 tons instead of 12.9 for the screw boat.

The speed of the hydraulically propelled boat was

$$\frac{12.6 \times 6080}{60 \times 60} = 21.3$$

feet per second, and the velocity of discharge was 37.2 feet per second. The acceleration imparted to the water was

$$37.2 - 21.3 = 15.9$$

feet per second. This gives for the ratio

$$\frac{s}{V+s} = \frac{15.9}{37.2} = 0.43,$$

which for a screw propeller corresponds to the apparent slip; the apparent slip is usually 0.10 to 0.15 for screw propellers.

Paddle-wheels.—Originally all steamships were propelled by paddle-wheels, but the tendency has been to restrict that method of propulsion to the navigation of shallow waters, and even for that purpose it is now found convenient to use two or more small propellers with a high rotative speed. Two types of wheels have been used, radial wheels and feathering wheels; the latter have the paddles or floats guided by a special mechanism in such a manner that there is less splashing as they enter and leave the water. As will appear in the description of this mechanism, the effect of feathering the paddles is usually equivalent to using a radial wheel having twice the diameter. Feathering wheels were early introduced on English steamers because they had to be seaworthy vessels without excessive height at the paddle-boxes, and further the comparatively small diameter and high speed of the wheels allowed the use of compact engines that could be conveniently stowed in the hull. On the other hand American steamboats were developed for traffic on rivers and sheltered waters making short trips. They were given large wheels

driven by vertical-beam engines which were cheap and effective, and the use of feathering paddles has been comparatively recent.

Feathering Paddle-wheels.—To avoid splashing a paddle should enter the water edgewise. In Fig. 175 let a represent the centre of a paddle-wheel, and WW the level of the water. Lay off ab equal to the velocity of the ship, and ac equal to the velocity of rotation of the middle of the paddle, at right angles to ab; then bc is the absolute motion of the paddle, and consequently the paddle should be parallel to bc in order that it may enter edgewise. This construction is made for the middle of the blade.

FIG. 175.

To apply this to a wheel, Fig. 176, let o be the centre of its axis and aa' be the water-line; make the construction abc and draw ae parallel to bc; make the corresponding construction at a', and

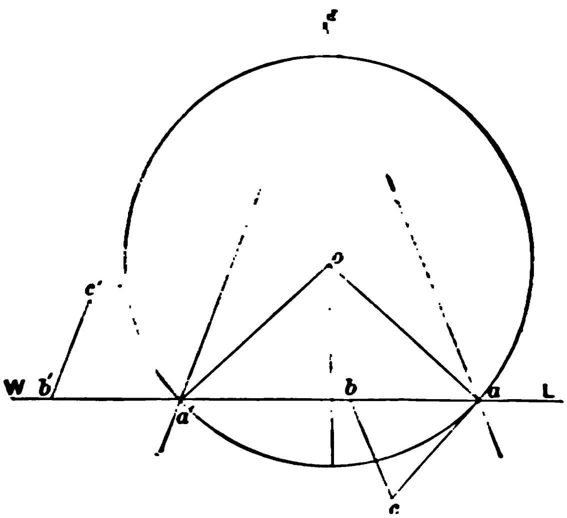

FIG. 176.

draw $a'e$ parallel to $b'c'$. To show that e is on the vertical through o it is sufficient to note that the angle $cba = bae$, and $c'b'a' = ba'e$. The

speed of the ship *ab* is less than *ac*, the speed of revolution of the paddle, and the construction places *e* above the circle as shown. If a similar construction is made with $ba = bc$, then *e* will fall on the circumference of the circle, for in such case $abc = acb = bae$, so that the isosceles triangles *abc* and *aea'* similar. But the angle *bac* is measured by half the arc *aa'*, and the angle at *e* is measured by half the same arc; that is, *e* is on the circle. It is customary to construct feathering paddle-wheels with the radiating point *e* on the circle; this is equivalent to using a radial wheel with double the diameter.

The feathering mechanism can be constructed as shown by Fig. 177, where *o* is the centre of the wheel and *WL* is the water-line. At *a*, *a'*, and *a''* draw three positions of a blade radiating from *e*, the end of the vertical diameter through *o*. Draw *bac* perpendicular

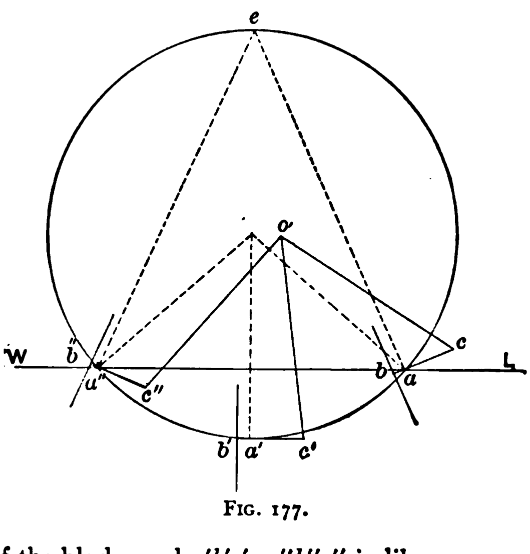

FIG. 177.

to the face of the blade, and *a'b'c'*, *a''b''c''* in like manner; the paddle is pivoted at *a* to the frame of the wheel, and is moved by a rod jointed to *o'*; construct a circle through the three points *c*, *c'*, and *c''* with its centre at *o'*; this locates the centre fixed to the hull or the guard, from which the guide-rods radiate. As there are a number of paddles all having guide-rods radiating from *o'*, it is customary to make one of them a master-rod which controls the other . The

master-rod is joined to a disk which is pivoted at o', and it is rigid enough to turn that disk; the other rods, which act in tension only, are pivoted to the perimeter of the disk. Commonly the paddles are guided at one end only, and then o' is fixed to the guard; some large paddle steamers have the paddles guided at both ends and then there must be a centre o' at both the guard and the hull; the latter must have for its pivot a disk large enough to include the paddle-shaft, which is centred at o.

Number and Dimensions of Paddles.—The restriction of the use of paddle-wheels and the stringency of the conditions of the service of paddle steamers, which are commonly passenger-boats that are required to have high speed on small displacement, has tended to throw the construction of such boats into the hands of a few builders who have accumulated information that has not been made public. The information concerning paddle steamers which is to be found in text-books does not appear to be recent, and the methods of determining dimensions for wheels are largely arbitrary.

If the mean diameter of the wheel measured to the middle of the paddles is D, and if R is the number of revolutions per minute, then the circumferential velocity of a paddle is

$$\frac{\pi DR}{60}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the speed of the ship is V knots per hour, then the velocity in feet per second is

$$\frac{6080V}{60\times60}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The apparent slip of the wheel is considered to be

$$s=\frac{\frac{\pi DR}{60}-\frac{6080V}{60\times60}}{\frac{\pi DR}{60}}=\frac{\pi DR-101.3V}{\pi DR}, \quad . \quad . \quad . \quad . \quad (3)$$

and this quantity is commonly assumed to be the acceleration imparted to the water in the discussion on page 424 of the fluid effi-

ciency of propulsion. Though there is but little direct evidence on this point, it is not improbable that this assumption is approximately true. From equation (3) the diameter may be computed provided the revolutions are properly selected, giving

$$D=\frac{101.3V}{\pi R(1-s)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The slip is likely to be from 15 to 25 per cent; and the revolutions may vary from 20 to 50 per minute, depending on the size of the vessel and the height at which the paddle-shaft can be conveniently placed.

The number of paddles will be determined by comparison with existing boats, and will depend largely on the general arrangement of the wheels. There may be one paddle for each foot of diameter of the wheel for radial wheels, and half as many for feathering wheels. The width of a paddle may be about one-third of the length.

The upper edge of the paddle may be immersed one-eighth of its breadth for a boat that navigates smooth waters, and one-half the breadth for rough waters. The thickness of a wooden paddle may be about one-tenth of the breadth, and the thickness of a steel paddle may be one-sixtieth of the breadth. Steel paddles are commonly curved to a radius equal to the diameter of the wheel to stiffen them and give a better hold on the water; they are somewhat less effective when backing.

In order to base the efficiency of paddle-wheels and the area of their paddles on the fluid efficiency, it may be assumed that the volume of water acted on in a minute is proportional to the continued product of the area of a paddle A, the diameter of the wheel D, and the number of revolutions per minute R. that is, to ADR; the mass of water acted on per second will be proportional to the same quantity. The acceleration per second imparted to the water may be assumed to be proportional to RDs, where s is the slip computed by equation (3). The propulsive force exerted by the paddles to overcome the resistance of the boat may be made proportional to the mass acted upon multiplied by the acceleration imparted, that is, to AR^2D^2s. The boat will move a distance proportional to $RD(1-s)$ in a minute; and the work applied per minute will be proportional to

this distance multiplied by the propulsive force. The work applied per minute is the power exerted by the paddle-wheels, which is proportional to the indicated horse-power of the engine, so that we may write

$$\text{I.H.P.} = \frac{AD^3R^3s(1-s)}{K}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and conversely the area of a single paddle is

$$A = \frac{K \times \text{I.H.P.}}{D^3R^3s(1-s)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The value of K is not well known, but may be assumed to vary from 500000 to 1000000. The following table gives the particulars of a few paddle-wheel steamboats with the indicated horse-power and values of the factor K.

PADDLE-WHEELS.

Name.	Speed. Knots.	Indicated Horse-power.	Area of a Paddle. Square Feet.	Diameter to Centres of Paddles. Feet.	Revolutions per Minute	Slip.	Factor K
1. Nantucket.	13.0	717	12.55	23.9	21.8	0.197	392000
2 Uncatena	13.0	902	19.56	18.9	29.0	0.234	640000
3 Gay Head	13.5	966	27.48	18.5	26.4	0.141	401000
4.	18.3	2520	34.00	16.4	41.0	0.155	537000
5.	18.3	2680	34.10	17.0	47.0	0.265	1264000
6 Tashmoo	18.8	3400	45.20	18.7	40.0	0.170	785000
7. City of Erie.	18.9	6472	48.00	24.5	33.3	0.250	755000

Of the boats named in the table the first three are small boats on the Atlantic coast, and the last two are boats on the Great Lakes; numbers 4 and 5 are English boats reported by Mr. Barnaby. The *Nantucket* is an old boat with radial wheels and flat paddles; all other boats in the table have feathering paddles; the paddles for number 5 were flat and made of wood; with the exception of number 1 and number 5, all the boats had steel paddles curved to a long radius.

Wave Contour.—The efficiency of a paddle-wheel depends to a great degree on the immersion of the paddles, and falls off rapidly when the immersion is too little or too great. But the water-line for

a boat which is driven at high speed is affected by the waves which accompany the boat, and also by the action of the wheel on the water. It is consequently difficult, if not impossible, to determine the real immersion of the paddles except by observation or experiment. The Dennys at Dumbarton have given much attention to tank experiments on paddle-wheel boats, both in determining the resistance and the proper location of the wheels, and have found great advantage therefrom.

Screw Propellers.—A screw or helical surface is generated by a line which moves forward uniformly and revolves uniformly with one point in contact with a line called the axis.

Fig. 178 represents one turn of a screw generated by a line at right angles to the axis *st*; the generating line starts at *sa* and moves to *tc*. A quarter-turn of the screw is represented by *uefgv*, and in end projection by $e'f'g'$. The helix, *abc*, can be developed by unrolling the cylinder on which it lies. A development of half of a helix is drawn from *f*; the base *fh* is equal to πr, representing the radius *sa* by r; the altitude $ih = \frac{1}{2}P =$ half the pitch. The angle α can be conveniently determined by laying off

$$lk = \frac{P}{2\pi}$$

and completing the triangle *flk*. The deviation between the helix from *f* to *g* and the line *fl* is small but appreciable; the deviation for $\frac{1}{18}$ of a turn from *f* to *n* is inappreciable.

If the generating line *sa* is inclined to the axis as in Fig. 179, the screw may be said to have a rake. A helicoidal surface may be generated by a curved line like *as*, Fig. 180.

Thus far it has been considered that the pitch is uniform as for a true screw; in dealing with screw propellers we have screws with variable pitch.

If the generating line as in Fig. 181 makes a constant angle with the axis, but moves forward with increasing velocity, the surface which it generates is called a screw with increasing axial pitch.

If the generating line makes a variable angle with the axis as in Fig. 182, the surface generated is said to be a screw with increasing

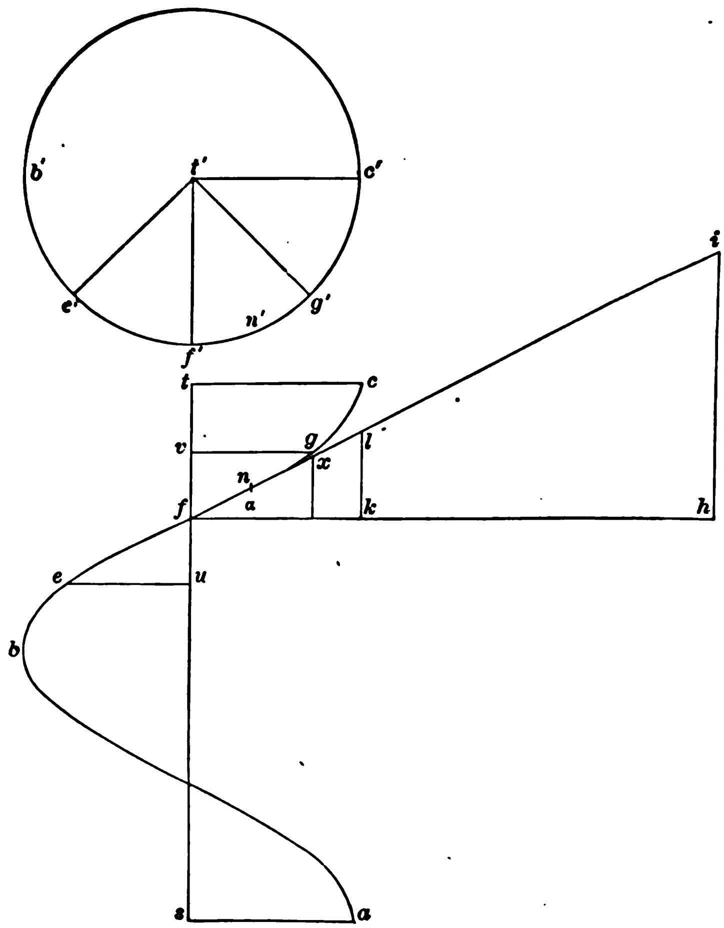

FIG. 178.

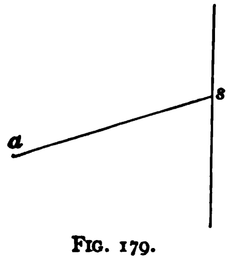

FIG. 179.

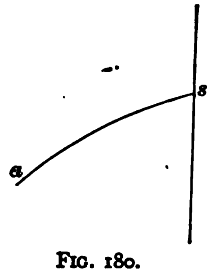

FIG. 180.

radial pitch; the pitch of the helix *ac* is clearly greater than that of the helix *eg*. There does not appear to be any advantage in using varying pitches for screw propellers, and consequently it is the common practice to use a true screw for the acting face; the back of the blade is rounded and is not a true screw. A screw with axially

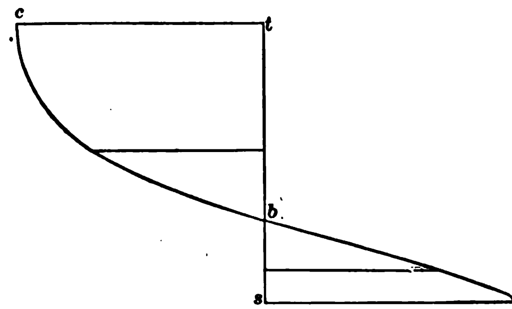

FIG. 181.

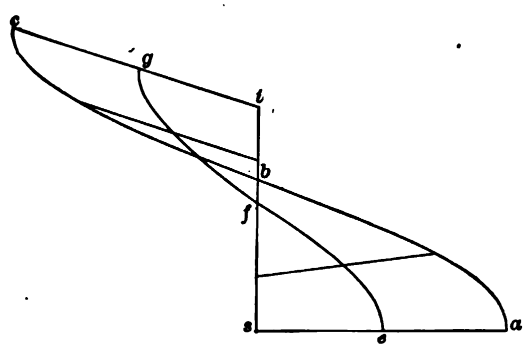

FIG. 182.

increasing pitch may be generated by a line at right angles to the axis as in Fig. 178, or the generating line may have a rake as in Fig. 179, or a curved line may be used as in Fig. 181. Radially increasing pitch will give a rake to part if not all of the screw. A curved line may be used with radially increasing pitch.

Propeller-blades extend to only a part of a turn of the helical surface, from $\frac{1}{8}$ to $\frac{1}{4}$ of a turn being commonly used. A propeller may have two, three, or four blades. If a ship has one propeller, it is commonly given four blades; a twin-screw ship has usually three

blades for each propeller; auxiliary yachts sometimes have two-bladed screws.

Development of Blades.—The surface of a screw propeller is a ruled surface which cannot be developed as a cylindrical or conical

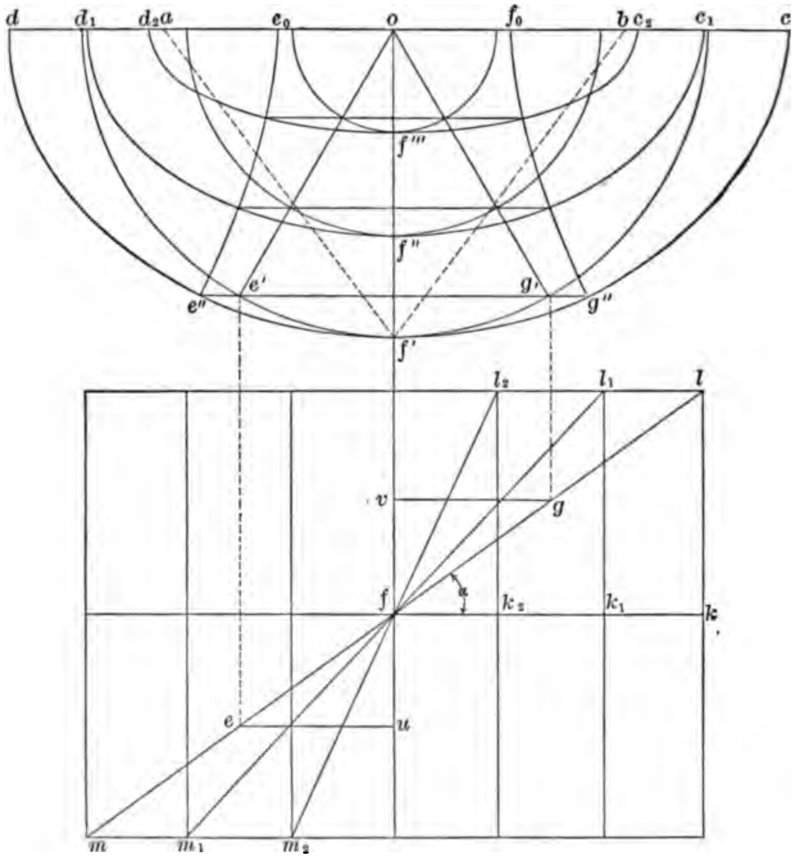

FIG. 183.

surface is developed, but there is a conventional method of getting a plane figure which has nearly the same surface as the blade; this method is called developing the blade, and the figure is called the developed contour of the blade.

In Fig. 183 let $uefgv$ be one projection of a part of a helical surface, and $e'Og'$ be the other projection; one-sixth of a turn of

the screw is taken as convenient for the present purpose. The helix from e to g is replaced by a straight line obtained by developing a part of the cylinder on which it lies, as in Fig. 178. At f the horizontal line fk is drawn equal to the radius of the cylinder, and at k the line lk is laid off equal to

$$\frac{P}{2\pi},$$

thus determining the angle α.

A plane passed through the line mfl perpendicular to the axis of the screw will cut an ellipse from the cylinder on which the helix lies; $e'f'g'$ represents a partial projection of the cylinder and of the helix lying on it, and also of the ellipse cut by the plane through mfl. An arc of the ellipse from e to g will deviate but little from the true helix and may be used instead of it. The ellipse may be revolved until it is perpendicular to the axis of the screw and will then be shown in its true form at $de''f'g''c$; the foci of the ellipse are at a and b, obtained by drawing an arc from f' with a radius equal to fl and intersecting cd at a and b. The triangle $of'b$ is equal to the triangle kfl; consequently b may be located by laying off

$$ob = \frac{P}{2\pi}.$$

The revolution of the ellipse will "develop" the blade at efg, $e'f'g'$ into the elliptical arc $e''f'g''$.

The same construction is made in the figure for two other points, f'' and f'''; the corresponding helices lie on cylinders $m_1l_1k_1$, $m_2l_2k_2$. Planes through m_1l_1 and m_2l_2 cut ellipses from these cylinders which are shown in true form at $c_1f''d_1$ and $c_2f'''d_2$. The foci of all the ellipses are at a and b, being located by laying off

$$ob = lk = l_1k_1 = l_2k_2.$$

At the axis the width of the blade is uv laid off at e_0f_0. The contour of the blade is $f_0g''f'e''e_0$. The developed blade contour has somewhat less area than the real helical surface, as is evident from the consideration that the straight line fx, Fig. 178, is shorter

than the helical curve from f to g; the difference is insignificant for all forms of propellers now in use.

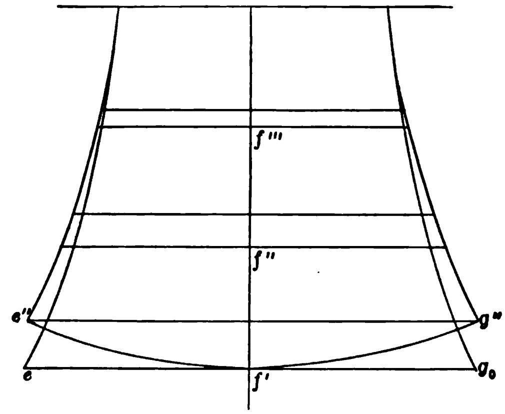

FIG. 184.

A simpler method of developing the blade is shown by Fig. 184. Here straight lines are drawn at f', f'', and f''', and on them are laid off the lines e_0g_0, etc., equal to efg, etc., from Fig. 183. The development by aid of elliptical arcs is superposed to show the difference in the result of the two different methods. The area obtained by this simpler method will be larger or smaller than that by the use of elliptical arcs, and will have a different distribution, depending on the form of the propeller-blade; this will be referred to again.

Early propellers had the blades shaped like that represented by Fig. 183, except that they were usually narrower and had rounded corners at e and f; the corner of the forward or the leading edge of the blade especially should be rounded or else the propeller is likely to give vibrations. This form is still used to a large extent wherever a large blade area is required, as for fast yachts, torpedo-boats, and tugboats.

The form of developed blade adopted by the British Admiralty is represented by Fig. 185. It is an ellipse which has its major axis equal to the radius of the wheel measured from the axis to the tip; its minor axis is $\frac{2}{5}$ of its major axis, that is, $\frac{2}{10}$ of the diameter of the wheel. The blades are made separate and are fixed to a spherical hub which has a diameter large enough for convenient construc-

tion, varying from .22 to .27 of the diameter of the wheel. The area of the complete ellipse is

$$\frac{\pi ab}{4}=\frac{\pi\times 0.2D\times 0.5D}{4}=0.1\frac{\pi D^2}{4},$$

where D is the diameter of the wheel. Allowance must of course be made for the portion of the blade cut off by the hub. It will be sufficient to treat the area cut off as a segment of the ellipse. Now an ellipse can be treated as the projection of a circle, and the area of its segment will have the same proportion to the area of the entire ellipse as the area of the circular segment of which it is the projection has to the area of the circle. So that the area of the segment can be obtained from a table of circular segments. For example, a hub with a diameter .22 of the diameter of the wheel will cut off a segment with a rise of .22 of the major diameter of the ellipse. From a table of circular segments it appears that a circular segment with a rise of .22 of the diameter has an area equal to 0.128 of that of the entire circle; it is, therefore, evident that the hub will cut off approximately 0.128 of the area of the blade, and the effective area will be

$$0.1-0.0128=0.0872$$

of the circle swept by a tip of the wheel.

Naval Constructor Taylor has proposed the form of blade represented by Fig. 186. The maximum width is made 0.2 of the diameter, as for the Admiralty blade, and the blade from the middle to the hub is elliptical; from the middle to the tip the contour of the blade is represented by the equation

$$\frac{x^3}{a^3}-\frac{y^3}{b^3}=1,$$

which makes the tip appreciably wider, as is evident from inspection of Figs. 185 and 186. The contour is readily laid out by the relative widths given on the dotted lines; the elliptical blade can be constructed in the same way if desired, using the figures on Fig. 185.

The average width of the blade in Fig. 186 from the outside of the hub to the tip is 0.924 of the maximum width; if this latter be

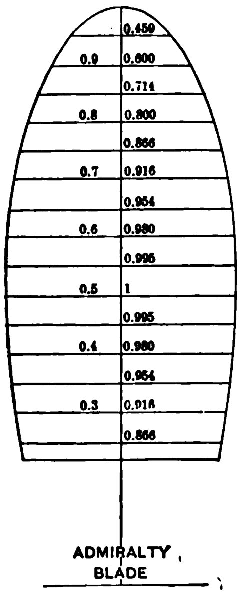

FIG. 185.

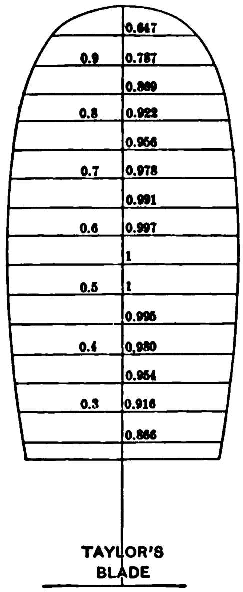

FIG. 186.

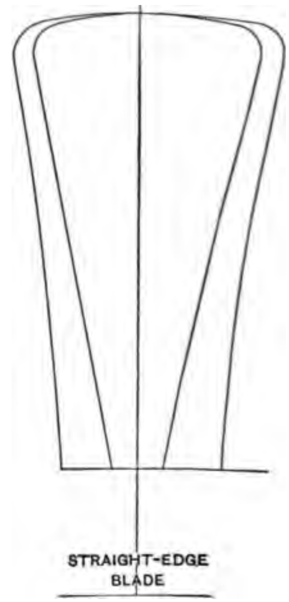

FIG. 187.

taken as $0.2D$ and if the diameter of the hub is $0.22D$, then the area of the blade is

$$0.924 \times 0.2D \times \tfrac{1}{2}(1-0.22)D = 0.072D^2,$$

which is

$$0.072D^2 \div \tfrac{1}{4}\pi D^2 = 0.091$$

of the disk area of the wheel. Taylor's blade is consequently about four per cent larger than the Admiralty blade, which has the same main dimensions.

If the straight-edged propeller blade is developed by the method of Fig. 184, and if the curved edges are replaced by straight lines, then the mean width of the blade will be very nearly

$$\frac{1}{2n}\{(\pi^2D^2+P^2)^{\frac{1}{2}}+(\pi^2d^2+P^2)^{\frac{1}{2}}\},$$

where D and P, as before, are the diameter and pitch of the wheel, while d is the diameter of the hub and $\frac{1}{n}$ is the fraction of a turn of the screw used. Fig. 187 represents the development of a straight-edged propeller-blade which has a pitch equal to twice the diameter. If the diameter of the hub is one-ninth of the diameter of the wheel, then the mean width by the above expression is .018 of the diameter and the area is about 0.09 of the area of the disk of the wheel. The blade has therefore about the same developed area as Taylor's blade, and the comparison of the three forms in Figs. 185, 186, and 187 will be found interesting and instructive.

Projections of a Propeller-wheel.—The projections of a straight-edged propeller-wheel can be drawn at once as in Fig. 183 and Fig. 187, and then the development can be constructed as in the former figure or by the simpler method of Fig. 184. The usual methods of designing and making the pattern of such propellers do not depend on the development, which is consequently seldom drawn. But when the form of the developed blade is taken as the basis of a design, it becomes necessary to construct projections for the use of the pattern-maker and for the general plans of the ship and machinery.

The usual method of drawing the projections of a propeller-wheel is shown by Figs. 188 and 189, where OG represents the middle line of a blade which is set square with the shaft. The blade is of the Admiralty form, and its development is represented by the dotted ellipse in Fig. 188. The projection of the blade on a plane perpendicular to the shaft is shown by Fig. 188; Fig. 189 shows the projection of the blade on a plane parallel to the shaft, and also the projection IOL on the same plane of another blade which is at right angles to the first blade.

Using the method of projection shown by Fig. 183, a point B is taken on the middle line of the blade, and through it there is drawn a circular arc BE from the centre O of the shaft; it may represent a part of a cylinder which has its axis coincident with the axis of the shaft. By the method shown this cylinder is to be cut by a plane passing through OB and through the tangent to a helix at B;

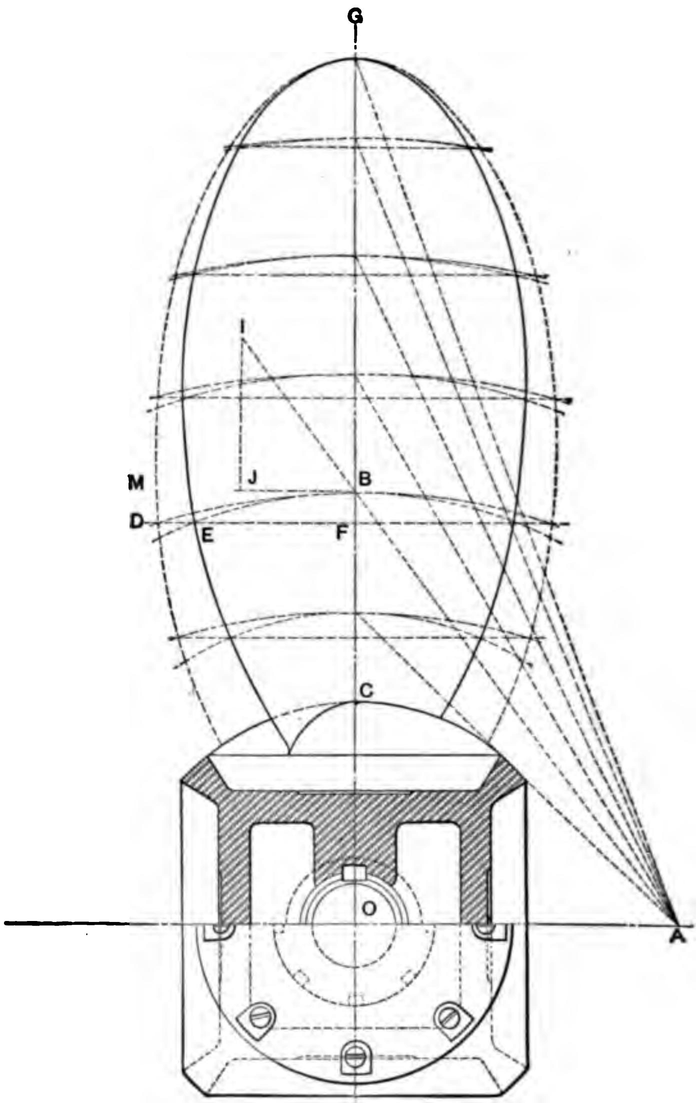

FIG. 188.

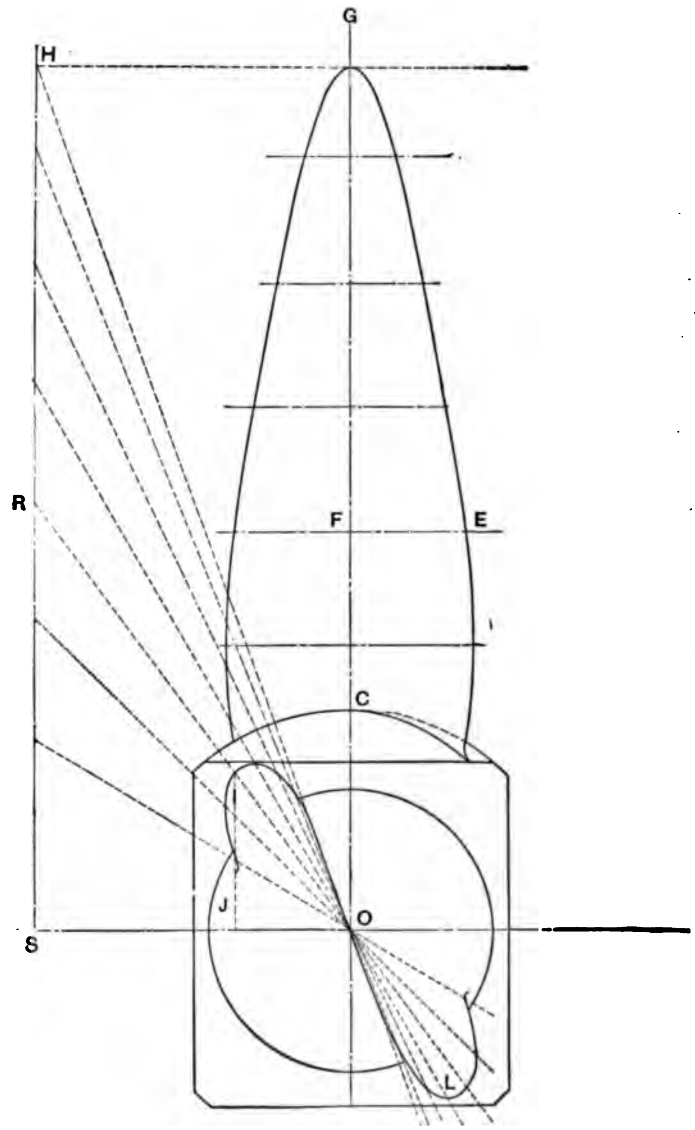

FIG. 189.

and the elliptical section is to be revolved into the plane of the paper. To find the focus of the ellipse lay off

$$OA = \frac{P}{2\pi}.$$

From the focus at A and another focus similarly placed on the other side of O draw the elliptical arc BD. When the plane containing the elliptical arc BD is revolved into its proper place the projection of the point D will travel along DE from D to E, which is a point on the projection of the blade. After a sufficient number of projected points like E are located the projected contour can be drawn as indicated. The middle line of a blade at right angles to OG is projected at the point O, Fig. 189, the projected contour being IOL. The line OR in that figure is the projection of the tangent at a point corresponding to the point B of Fig. 188; the angle ROS is determined by making

$$OS = OA = \frac{P}{2\pi},$$

and by making SR equal to OB. This line is also the trace of the plane passing through the tangent to the helix and the middle line the blade. The half-elliptical chord FD, Fig. 188, may here be laid off in its true length at IO and OL to determine two points of the contour IOL. The half-breadth of the blade may be projected to FE to determine a point of the contour of the blade OG. It is also evident that IJ is equal to the projection of the half-chord ED of Fig. 188 on the plane at right angles to the shaft, or, in other words, IJ is equal to FE, Fig. 188, and may be used for locating E more exactly than by the intersection of the line DF by the circular arc BD. If desired, an equivalent construction can be made by laying off BI in Fig. 188 equal to FD and then drawing the horizontal and vertical lines BJ and JI; the latter, JI, may be laid off at FE to locate E, and the former at FE, Fig. 189.

If the approximate development like that of Fig. 184 is preferred, the construction just given may be made, producing the triangle BIJ. The half-breadth IJ may then be laid off on a horizontal line through

B, and the half-breadth *BJ* may be laid off in Fig. 189 at the sa
distance from the axis of the shaft. The effect of this constructi

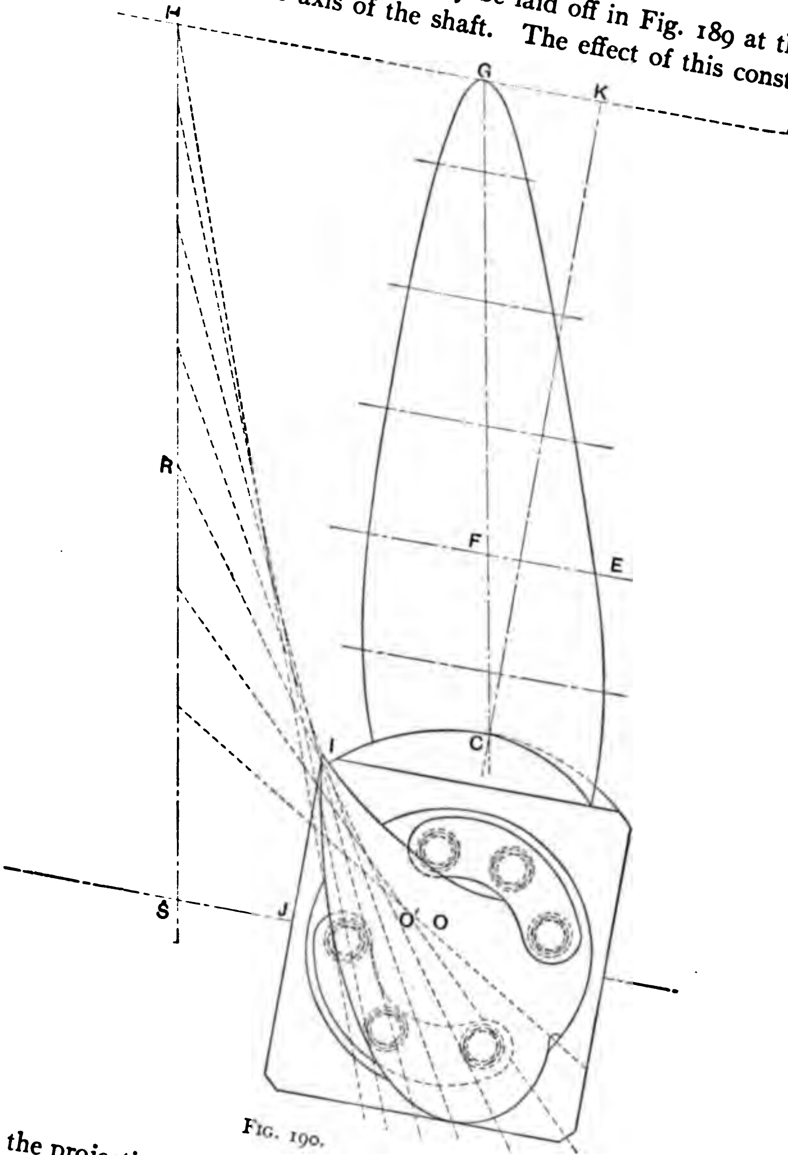

Fig. 190.

make the projections slightly fuller at the point of the blade.
the blade has a rake as in Fig. 190, the middle line of the blade

makes an angle such as *GCK* with the vertical line *KC*. The projection on a transverse plane is not affected, but is still represented

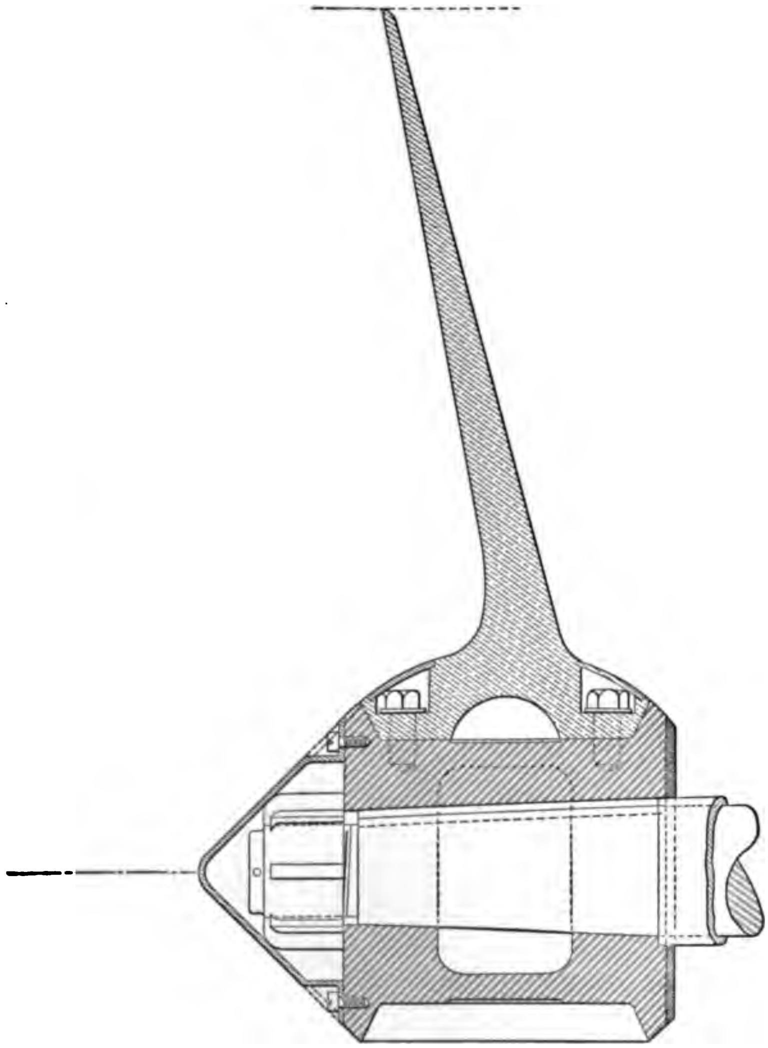

FIG. 191.

by Fig. 188. The projection of the same blade on a plane parallel to the shaft is obtained by laying off the half-widths, like *EF*, Fig. 189, from the inclined line *GC*. To get the end projection of the other

blade, draw RS in Fig. 190 parallel to GC, and in place of O take O', the projection of F on OS; draw the line $O'R$; since both R and O' are shifted to the left the same distance the line $O'R$ makes the same angle with OS as does OR of Fig. 189. The half-breadth $O'I$ is now made equal to DF of Fig. 188.

Fig. 191 gives a section of a blade by a plane through the shaft and the middle of the blade, which here has a rake. Fig. 192 gives the sections of the blade by planes at DF and corresponding places. The four outer sections are flat on the acting face and are rounded on the back; the inner section is slightly rounded on the face also. The faces of the sections are in reality slightly curved, since they are sections of a helicoidal surface, and where such sections are used in making a pattern or in moulding a blade the correct curve must be given to them. Fig. 193 shows the same sections arranged as they would appear for the blade projected at IOL, Fig. 189, without any rake.

The projection IOJ on Fig. 189 shows the edge of the blade; the projection of the same blade on Fig. 190 allows for the thickness of the blade and gives the projection of the solid form of the blade.

This propeller has four blades, of which two only are represented to avoid confusion; if the edge of the blade is only considered, the blade below the shaft has the same form as that above; but if the thickness of the metal is considered, the forms are slightly different. The intersection of the blade with the spherical surface of the hub is commonly represented by a curve as in Figs. 188 and 189; in reality the intersection is not sharp because the corners are filled in as indicated by Fig. 192. As nothing depends on the representation of the intersection, it may be drawn in free hand; at most one or two points between C and the edge of the blade on Fig. 188 need be located.

On the general plans of ships it is customary to draw projections of all the blades of a propeller for the sake of pictorial effect; on the drawings of the propelling machinery the projections and sections of one blade are sufficient. The projections of a three-bladed propeller are commonly drawn with one blade erect and with the other blades at 180° from it. The projection on a plane per-

pendicular to the shaft shows three identical projections of the blades; the projection on a plane parallel to the shaft commonly

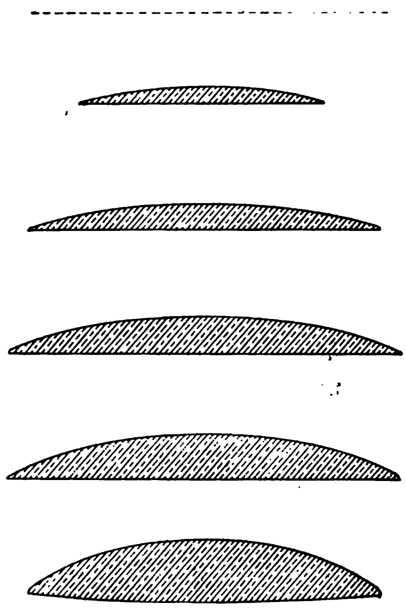

FIG. 192.

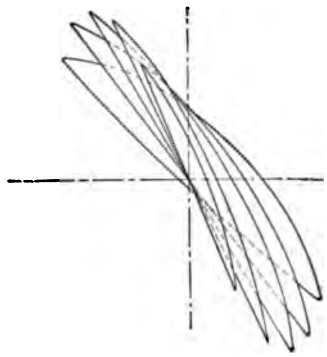

FIG. 193.

shows one blade, as in Fig. 188, and one below the shaft that can be constructed by the usual methods of projection.

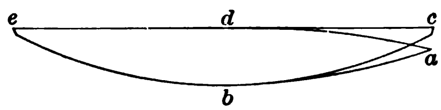

FIG. 194.

The aftermost or driving face of the propeller-blade is commonly made a true helical surface unless, as in Fig. 192, that face is some-

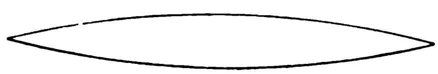

FIG. 195.

what rounded to avoid too much protuberance at the back. The edges especially of cast-iron propellers are usually thickened as indicated at *c* and *e* of Fig. 194 by drawing small circular arcs. When

the thickness is all applied to the back of a blade an eddy may be formed just aft of the leading edge, such as *c*, Fig. 194; to avoid such an action that edge is sometimes turned back or "refused," as indicated by *dab*. The usual habit of applying all the thickness to the back of the propeller probably comes in part from an unexpressed idea that the face (after side) of the blade is alone concerned in propulsion, and in part from the fact that it is convenient to have one true helical surface on the pattern. It is, however, probable that the back of the blade has quite as large an influence as the face, and that a more logical construction would divide the thickness between the face and the back, as in Fig. 195; such is the habit of Mr. N. G. Herreshoff.

Froude's Experiments.—Our systematic knowledge of the action of screw propellers is founded mainly on tests made by the Froudes,* those by Wm. Froude on small flat plates and those by R. E. Froude on model propellers. Experiments on models were made also by Thornycroft both for screw propellers and for turbine propellers, but they are less extensive and less complete than Froude's experiments.

The experiments by Froude on models were made in a towing-tank. The propeller models were all of the same diameter, namely, 8.16 inches; four pitches were used, defined by the ratios 1.225, 1.4, 1.8, and 2.2 of the pitch to the diameter; tests were also made with two, three, and four blades. The developed contour of the blade was an ellipse as shown by Fig. 185, page 440, for the standard Admiralty blade.

The model was mounted on a frame depending from the towing-car and was arranged so that the thrust of the screw and the moment required to turn it could be measured. The car could, of course, be towed at various speeds. and the screw could be revolved at desired rates. The distance that the screw would run in a minute if it were working in a fixed nut is evidently PR, where P is the pitch in feet and R is the number of revolutions per minute; if the car traverses v feet per second, the slip is

$$s=\frac{\frac{1}{60}PR-v}{\frac{1}{60}PR}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Proc. Inst. Naval Archts., vols. xix, xxiv. and xxvii.

If the diameter of the wheel from tip to tip is D feet, then the pitch ratio is

$$p = P \div D. \qquad (2)$$

It was found that even when there was no slip by equation (1) there might be some thrust of the screw, and that the thrust became zero only for a negative slip. But as the screw was a true helical surface on the acting face with the thickness of the blade all applied at the back, this action is properly attributed to the fact that we are dealing with a modified helical surface.

It was found that a three-bladed screw was slightly more efficient than a four-bladed crew, and that a two-bladed screw was even more efficient; the differences, however, are not large and are usually neglected in designing screws from Froude's data. It was, however, found that three blades gave more than three-fourths of the thrust of four blades of the same form, and in like manner that two blades gave more than half as much thrust; thus if the thrust of a four-bladed screw be called 1, then a three-bladed screw will be 0.865, and a two-bladed will be 0.65.

If the thrust of the screw is T pounds and if the speed of the screw is v feet per second, then the work delivered by the screw is Tv foot-pounds per second; if M is the turning moment applied to the propeller-shaft through a dynamometer, then the work per second supplied to the screw is

$$\tfrac{1}{60}M 2\pi R,$$

and the efficiency of the screw is

$$\frac{60Tv}{2\pi RM}. \qquad (3)$$

When the thrust is zero the efficiency is also zero; this condition, as already said, is found for a negative slip calculated by equation (1), considering only the pitch of the acting helical face of the blades. As the slip increases the efficiency also increases and soon reaches a maximum; slips greater than that which give the maximum efficiency give a slowly decreasing efficiency as the slip increases.

The form of curves of efficiency is indicated by Fig. 196, which represents certain experiments made by Professor Durand. The following table gives the slips which give maxima for efficiencies with various pitch ratios; all for the Admiralty blade having a width equal to two-tenths of the diameter of the wheel.

PITCH RATIO AND SLIP FOR MAXIMUM EFFICIENCY.

Pitch Ratio.	Real Slip of Screw.	Pitch Ratio.	Real Slip of Screw.
0.8	15.55	1.7	21.3
0.9	16.22	1.8	21.8
1.0	16.88	1.9	22.4
1.1	17.55	2.0	22.9
1.2	18.20	2.1	23.5
1.3	18.80	2.2	24.0
1.4	19.50	2.3	24.5
1.5	20.10	2.4	25.0
1.6	20.70	2.5	25.4

Durand's Experiments.—Experiments were made by Prof. Durand in much the same way on model propellers with varying width of blade. His experiments were made on models on a shaft protruding from the bow of a small yacht which steamed at determined speeds over a measured course. His propellers all had four blades and were of the Admiralty type except that the minor diameter of the ellipse varied from 0.1 to 0.4 of the diameter of the wheel, that is, from half to twice the standard width, which is 0.2 of the diameter. The models were all one foot in diameter and had a pitch ratio of 1.3; the thickness of the blades was applied to the back of the blade only. Seven widths of blade were used, varying uniformly from the minimum width of 0.1 to the maximum of 0.4 of the diameter of the wheel.

The results of these tests can be best summed up from the curves of efficiency given by Fig. 196. Each efficiency curve is numbered at the right with the width of the blade expressed in *tenths of the radius* of the wheel. The following facts will be noted:

(1) The narrowest blade gives zero efficiency for zero slip.

(2) All the blades give the same efficiency for 20 per cent slip.

(3) Wider blades give considerable efficiency and consequently considerable thrust for zero slips; the thrust and efficiency increase with the width, but not proportionally.

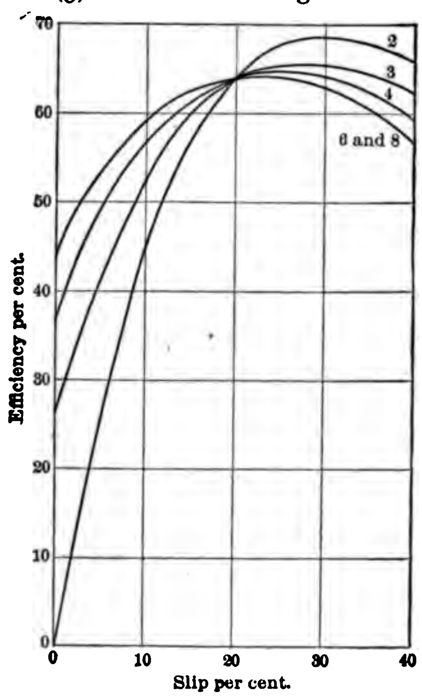

FIG. 196.

(4) The narrowest blade has a somewhat greater maximum efficiency than the other blades, the advantage being about five per cent.

(5) A wide blade reaches its maximum efficiency with a comparatively small slip; conversely, a narrow blade reaches its maximum efficiency only with a large slip. This conclusion applies only to four-bladed screws of the given pitch ratios (1.3) and is not in contravention of the table on the preceding page.

(6) It will be noted that the wider blades, 6 and 8, give substantially identical results; from this and from other facts brought out in the paper it appears that there is no advantage in increasing the width of the blade beyond 0.6 of the radius or 0.3 of the diameter, that is, beyond one and a half times the standard width.

From these tests it appears that (for the pitch ratio 1.3) the efficiency of a propeller at 20 per cent slip is independent of the width of the blade; the thrust increases with but more slowly than the width. At 20 per cent slip the thrust for a width of .3 diameter is only 1.1 of that for 0.2 diameter; while the thrust for .1 diameter is .55 of that for 0.2 diameter.

Froude's Theory.—The theory of the screw propeller which will be developed here is the blade theory, which was first propounded by Wm. Froude;* the method chosen is one given by Naval Con-

* Proc. Inst. Naval Archts., Vol. xix, p. 47

structor Taylor,* and applied by him to the practical design of propeller with some minor changes.

In Fig. 197 let AD be the development of one turn of the helix passing through an area ΔA of a helical surface. Then

$$AB = \pi D_a,$$

where D_a is the diameter of the cylinder on which the helix through ΔA is constructed; and

$$BD = P = \text{pitch}.$$

If the slip of the propeller is s, then the distance that the area ΔA will move for one turn parallel to the axis of the propeller is

$$P(1-s) = BC.$$

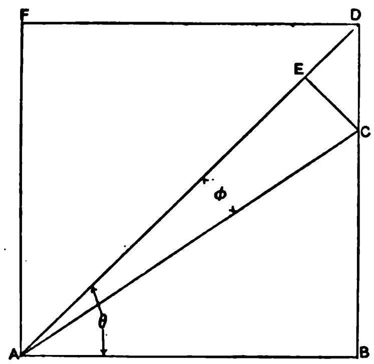

FIG. 197.

The actual motion for one turn is

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{\pi^2 D_a^2 + P^2(1-s)^2}, \quad . \quad . \quad . \quad (1)$$

and if the screw makes R revolutions per minute, the velocity of the small area ΔA in feet per second can be obtained by multiplying the expression just written by $\frac{1}{60}R$.

From certain experiments on thin flat plates which will be given on page 547 in connection with a discussion of rudders, Froude found that the normal pressure of the water on a plate moving through it was represented by the expression

$$P_n = 1.7Av^2 \sin i,$$

where P_n is the normal pressure in pounds, A is the area in square feet, v is the velocity in feet per second, and i is the angle which the plane makes with the direction of motion; 1.7 is a constant which may be replaced by the letter a for the present purpose. Applying this equation to the small helical area ΔA, its normal pressure may be written

$$\Delta N = \frac{1}{60^2} a \cdot \Delta A \cdot R^2[\pi^2 D_a^2 + P^2(1-s)^2] \sin \phi, \quad . \quad . \quad (2)$$

* Resistance of Ships and Screw Propulsion.

where ϕ is the angle EAC, Fig. 197, which the plane ΔA makes with its path AC.

The velocity of the water gliding past the plane ΔA will be greater than the velocity of the latter on the path AC; it is convenient to assume that the velocity of the water is equal to

$$\tfrac{1}{60}R \cdot AD = \tfrac{1}{60}R\sqrt{\pi^2 D_a^2 + P^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Making use of Froude's experiments on the friction of water on plane surfaces (see page 404), we have for the frictional resistance of the area ΔA

$$\Delta F = \frac{1}{60^2} f \cdot \Delta A \cdot R^2(\pi^2 D_a^2 + P^2). \quad . \quad . \quad . \quad . \quad (4)$$

Fig. 198 gives the diagram of forces acting on the area ΔA; the normal resistance is laid off at Ah perpendicular to the plane ΔA in such a direction that it acts as a force urging the plane upwards; the friction is laid off along the plane as a resistance to the motion of the plane. The two forces ΔN and ΔF are resolved into vertical and horizontal components; the former act as propelling force (and resistance), and the latter gives the basis for the calculation of the turning moment that must be applied to the propeller-shaft. The propelling force is

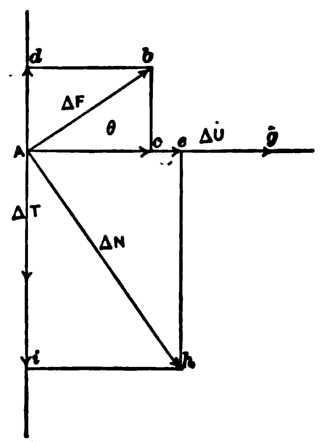

Fig. 198.

$$\Delta T = \Delta N \cos\theta - \Delta F \sin\theta, \quad . \quad . \quad (5)$$

and the turning force is

$$\Delta U = \Delta N \sin\theta + \Delta F \cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The thrust or propelling force ΔT will in one turn act through the distance BC (Fig. 197), and if the propeller makes R turns per minute will produce in one second the work

$$\Delta W_t = \Delta T \times \tfrac{1}{60}R \times BC = \tfrac{1}{60}PR(1-s)\Delta T. \quad . \quad . \quad . \quad (7)$$

The turning force ΔU will move through the distance

$$\frac{\pi}{60}D_aR$$

in one second, so that the work applied in one second will be

$$\Delta W_u=\frac{\pi}{60}D_aR\Delta.U. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In order to develop the expressions for the useful work ΔW_t and the gross work ΔW_u it is necessary to replace ΔT and ΔU by their values given in equations (5) and (6), and further to introduce the values of ΔN and ΔF from equations (2) and (4).

As for the trigonometric functions, it appears from Fig. 197 that

$$\sin\theta=\frac{BD}{AD}=\frac{P}{\sqrt{\pi^2D_a^2+P^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\cos\theta=\frac{AB}{AD}=\frac{\pi D_a}{\sqrt{\pi^2D_a^2+P^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\sin\phi=\frac{CE}{AC}=\frac{CD\cos\theta}{\sqrt{\pi^2D_a^2+P^2(1-s)^2}}$$
$$=\frac{BDs\cos\theta}{\sqrt{\pi^2D_a^2+P^2(1-s)^2}}=\frac{Ps\cos\theta}{\sqrt{\pi^2D_a^2+P^2(1-s)^2}}. \quad . \quad . \quad . \quad (11)$$

It is convenient to introduce the term diameter ratio, by which is meant the ratio of the diameter to the pitch; the diameter ratio is for the area ΔA and is not that for the tip of the propeller-blade. Assuming, then,

$$d_a=D_a\div P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and eliminating both D_a and P from the above trigonometric functions, we have

$$\sin\theta=\frac{1}{\sqrt{\pi^2d_a^2+1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\cos\theta=\frac{\pi d_a}{\sqrt{\pi^2d_a^2+1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\sin \phi = \frac{s \cos \theta}{\sqrt{\pi^2 d_a^2 + (1-s)^2}} = \frac{s\pi d_a}{\sqrt{\pi^2 d_a^2 + 1}\sqrt{\pi^2 d_a^2 + (1-s)^2}}. \qquad (15)$$

Making the proper substitutions in equations (7) and (8),

$$\Delta W_t = \tfrac{1}{60} PR(1-s)\left\{\frac{1}{60^2}a \,.\, \Delta A \,.\, R^2[\pi^2 D_a^2 + P^2(1-s)^2] \sin\phi \cos\theta \right.$$

$$\left. - \frac{1}{60^2} f \,.\, \Delta A \,.\, R^2[\pi^2 D_a^2 + P^2] \sin\theta \right\}.$$

$$\therefore\ \Delta W_t = \frac{1}{60^3} P^3 R^3 (1-s) \Delta A \left\{ \frac{as[\pi^2 d_a^2 + (1-s)^2] \pi d_a \pi d_a}{\sqrt{\pi^2 d_a^2 + 1}\sqrt{\pi^2 d_a^2 + (1-s)^2}\sqrt{\pi^2 d_a^2 + 1}} \right.$$

$$\left. - f \frac{\pi^2 d_a^2 + 1}{\sqrt{\pi^2 d_a^2 + 1}} \right\};$$

$$\therefore\ \Delta W_t = \frac{1}{60^3} P^3 R^3 (1-s) \Delta A \left\{ \frac{as\pi^2 d_a^2 \sqrt{\pi^2 d_a^2 + (1-s)^2}}{\pi^2 d_a^2 + 1} \right.$$

$$\left. - f\sqrt{\pi^2 d_a^2 + 1} \right\}. \qquad (16)$$

$$\Delta W_u = \tfrac{1}{60} \pi P R d_a \Delta A \left\{ \frac{1}{60^2} a \,.\, R^2[\pi^2 D_a^2 + P^2(1-s)^2] \sin\phi \sin\theta \right.$$

$$\left. + \frac{1}{60^2} f \,.\, R^2[\pi^2 D_a^2 + P^2] \cos\theta \right\}.$$

$$\therefore\ \Delta W_u = \frac{1}{60^3} P^3 R^3 \Delta A \left\{ \frac{as[\pi^2 d_a^2 + (1-s)^2] \pi d_a \pi d_a}{\sqrt{\pi^2 d_a^2 + 1}\sqrt{\pi^2 d_a^2 + (1-s)^2}\sqrt{\pi^2 d_a^2 + 1}} \right.$$

$$\left. + f \frac{(\pi^2 d_a^2 + 1)\pi d_a \pi d_a}{\sqrt{\pi^2 d_a^2 + 1}} \right\};$$

$$\therefore\ \Delta W_u = \frac{1}{60^3} P^3 R^3 \Delta A \left\{ \frac{as\pi^2 d_a^2 \sqrt{\pi^2 d_a^2 + (1-s)^2}}{\pi^2 d_a^2 + 1} \right.$$

$$\left. + f\pi^2 d_a^2 \sqrt{\pi^2 d_a^2 + 1} \right\}. \qquad (17)$$

For convenience the functions which depend only on the diameter ratio and the slip may be replaced by single letters as follows:

$$\alpha_1 = \frac{\pi^2 d_a^2 \sqrt{\pi^2 d_a^2 + (1-s)^2}}{\pi^2 d_a^2 + 1}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

$$\beta = \sqrt{\pi^2 d_a^2 + 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

$$\gamma = \pi^2 d_a^2 \sqrt{\pi^2 d_a^2 + 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

In the application of the theory the slip can commonly be made 0.2 or nearly that; it is therefore convenient to use, instead of (18), another constant,

$$\alpha = \frac{\pi^2 d_a^2 \sqrt{\pi^2 d_a^2 + .64}}{\pi^2 d_a^2 + 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

On page 458 is a table of values of the functions α, β, and γ for all diameter ratios up to 1.3, advancing by hundredths.

Using these constants, we have, instead of equations (16) and (17), for

Useful work,

$$\Delta W_t = \frac{1}{60^3} P^3 R^3 (1-s)(as\alpha_1 - f\beta)\Delta A, \quad . \quad . \quad . \quad . \quad (22)$$

or

$$\Delta W_t = \frac{1}{60^3} P^3 R^3 (1-s)(as\alpha - f\beta)\Delta A. \quad . \quad . \quad . \quad . \quad (23)$$

Gross work,

$$\Delta W_u = \frac{1}{60^3} P^3 R^3 (as\alpha_1 + f\gamma)\Delta A, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

or

$$\Delta W_u = \frac{1}{60^3} P^3 R^3 (as\alpha + f\gamma)\Delta A. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The efficiency for the elementary area is

$$e = \frac{\Delta W_t}{\Delta W_u} = (1-s)\frac{as\alpha_1 - f\beta}{as\alpha_1 + f\gamma}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

or

$$e = (1-s)\frac{as\alpha - f\beta}{as\alpha + f\gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The values assigned to the coefficients of thrust and friction by Wm. Froude for a thin isolated plane, are

$$a = 1.7, \quad f = .0085,$$

the speed being in feet per second and the force in pounds. In order to allow for friction of both sides of the plane, f is taken at

FUNCTIONS FOR ELEMENTS OF SCREW-PROPELLER BLADES.

α

Diameter Ratio, $D_a \div P$.	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	00.8	0.09
0.1	0.077	0.092	0.110	0.128	0.148	0.168	0.190	0.213	0.237	0.262
0.2	0.286	0.314	0.341	0.369	0.398	0.427	0.449	0.488	0.518	0.540
0.3	0.581	0.614	0.645	0.678	0.711	0.744	0.777	0.810	0.844	0.878
0.4	0.912	0.946	0.980	1.014	1.048	1.083	1.117	1.151	1.185	1.220
0.5	1.254	1.289	1.323	1.358	1.392	1.426	1.461	1.495	1.529	1.564
0.6	1.598	1.632	1.667	1.701	1.734	1.769	1.803	1.837	1.871	1.905
0.7	1.939	1.973	2.007	2.041	2.075	2.109	2.143	2.177	2.211	2.245
0.8	2.277	2.311	2.344	2.378	2.412	2.445	2.478	2.512	2.545	2.578
0.9	2.612	2.647	2.678	2.711	2.745	2.778	2.813	2.846	2.879	2.911
1.0	2.944	2.978	3.012	3.043	3.075	3.108	3.141	3.174	3.207	3.240
1.1	3.273	3.306	3.339	3.372	3.405	3.437	3.469	3.502	3.535	3.568
1.2	3.601	3.633	3.666	3.698	3.730	3.763	3.796	3.829	3.861	3.892

β

0.1	1.048	1.058	1.068	1.080	1.092	1.106	1.119	1.133	1.148	1.164
0.2	1.181	1.198	1.216	1.233	1.252	1.271	1.290	1.310	1.331	1.353
0.3	1.374	1.395	1.418	1.440	1.463	1.486	1.509	1.533	1.558	1.581
0.4	1.605	1.630	1.655	1.680	1.706	1.731	1.757	1.783	1.809	1.836
0.5	1.862	1.888	1.915	1.942	1.969	1.996	2.023	2.050	2.078	2.106
0.6	2.134	2.161	2.189	2.217	2.245	2.273	2.301	2.330	2.358	2.386
0.7	2.416	2.444	2.473	2.501	2.530	2.559	2.588	2.617	2.646	2.675
0.8	2.705	2.734	2.763	2.792	2.821	2.851	2.880	2.910	2.939	2.969
0.9	2.999	3.028	3.058	3.087	3.118	3.147	3.177	3.207	3.236	3.267
1.0	3.297	3.327	3.357	3.387	3.417	3.447	3.477	3.507	3.567	3.567
1.1	3.597	3.627	3.658	3.688	3.718	3.748	3.779	3.809	3.839	3.869
1.2	3.899	3.930	3.960	3.990	4.021	4.052	4.083	4.113	4.144	4.176

γ

0.1	0.103	0.126	0.151	0.180	0.211	0.245	0.282	0.323	0.367	0.414
0.2	0.465	0.521	0.580	0.643	0.711	0.784	0.860	0.942	1.026	1.120
0.3	1.220	1.323	1.433	1.550	1.669	1.796	1.930	2.071	2.220	2.373
0.4	2.534	2.704	2.881	3.066	3.259	3.459	3.669	3.887	4.113	4.351
0.5	4.594	4.846	5.110	5.383	5.666	5.959	6.261	6.573	6.899	7.235
0.6	7.582	7.936	8.304	8.684	9.075	9.478	9.892	10.322	10.761	11.211
0.7	11.684	12.159	12.652	13.154	13.673	14.206	14.753	15.313	15.888	16.476
0.8	17.086	17.703	18.336	18.983	19.642	20.329	21.022	21.738	22.462	23.210
0.9	23.976	24.747	25.545	26.351	27.191	28.031	28.897	29.781	30.673	31.602
1.0	32.540	33.496	34.470	35.464	36.476	37.507	38.558	39.628	40.717	41.826
1.1	42.956	44.105	45.277	46.477	47.689	48.920	50.187	51.461	52.757	54.074
1.2	55.413	56.788	58.172	59.577	61.020	62.470	63.975	65.479	67.005	68.588

twice the value that would be used for the friction of the skin of a ship. It is interesting to note that, from the table on page 405, the resistance of one square foot of varnished surface is (length 2 feet) 0.41 for a velocity of 600 ft. per minute or 10 feet per second; the exponent being 2, the resistance at 10 feet velocity is 100 times that for one foot velocity, so that for the conventions given above the coefficient is

$$2 \times \tfrac{1}{100} \times 0.41 = 0.0082.$$

Application to Propeller-wheels. — Let Fig. 199 represent a developed blade similar to Fig. 185, page 440. A strip across the blade at the distance r from the axis will have the length l across the blade and the width Δr; its area will be

$$l\Delta r. \quad . \quad . \quad . \quad . \quad (28)$$

In order that our work may be readily tabulated for use with all sizes of wheels which have the same blade contour, it is convenient to replace l and Δr by $\frac{l}{D}$ and $\frac{\Delta r}{D}$ or $\Delta\left(\frac{r}{D}\right)$, where D is the diameter of the wheel. At the same time we may allow for the number of blades on the wheel by multiplying by N, making

$$\Delta A = ND^2\frac{l}{D}\Delta\left(\frac{r}{D}\right). \quad . \quad . \quad (29)$$

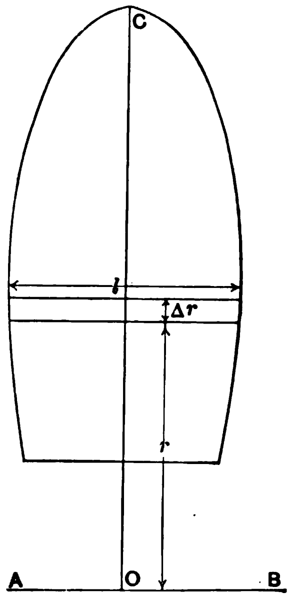

FIG. 199.

Replacing ΔA in equations (23) and (25) by this value and integrating,

$$W_t = \frac{1}{60^3}P^3R^3D^2N(1-s)\left[as\int \alpha\frac{l}{D}\,d\left(\frac{r}{D}\right) - f\int \beta\frac{l}{D}\,d\left(\frac{r}{D}\right)\right], \quad (30)$$

$$W_u = \frac{1}{60^3}P^3R^3D^2N\left[as\int \alpha\frac{l}{D}\,d\left(\frac{r}{D}\right) + f\int \gamma\frac{l}{D}\,d\left(\frac{r}{D}\right)\right]. \quad . \quad . \quad (31)$$

The integrals in these equations are most readily obtained graphically, as will be explained later and may be represented by single letters; thus,

$$A=\int \alpha \frac{l}{D} d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$B=\int \beta \frac{l}{D} d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$C=\int \gamma \frac{l}{D} d\left(\frac{r}{D}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Equations (30) and (31) will consequently be written:
Useful work of propeller,

$$W_t=\frac{1}{60^3}P^3R^3D^2N(1-s)(asA-fB); \quad . \quad . \quad . \quad (35)$$

Gross work for propeller,

$$W_u=\frac{1}{60^3}P^3R^3D^2N(asA+fC). \quad . \quad . \quad . \quad . \quad . \quad (36)$$

These amounts of work are in terms of foot-pounds per second. The letters have the following significance:

R = revolutions per minute;
N = number of blades;
P = pitch in feet;
D = diameter, tip to tip of blades, in feet;
s = slip (a decimal);
a = thrust factor;
f = friction factor;
A, B and C = functions depending on diameter ratio and contour of blades.

If desired, the performance of a propeller can be stated in horse-power by dividing the equations (35) and (36) by 550; from the useful work we shall by that means get the effective horse-power applied by the propeller to propelling the ship; from the gross

work we shall get the gross power which the engine must exert on the propeller-shaft:

$$\text{E.H.P.}=\frac{1}{550\times 60^3}P^3R^3D^2N(1-s)(asA-fB), \quad . \quad . \quad (37)$$

$$\text{G.H.P.}=\frac{1}{550\times 60^3}P^3R^3D^2N(asA+fC). \quad . \quad . \quad . \quad . \quad (38)$$

In any case the efficiency of the propeller will be

$$e=\frac{W_t}{W_u}=\frac{\text{E.H.P.}}{\text{G.H.P.}}=(1-s)\frac{asA-fB}{asA+fC}. \quad . \quad . \quad . \quad (39)$$

It will be noted that equation (39) differs from equation (27) in that α, β, and γ, which apply to an element of the surface, are replaced by A, B, and C, which apply to the entire wheel.

Equation (39) makes the efficiency of a propeller-wheel depend on the diameter ratio,

$$d=D\div P,$$

on the slip, and on the coefficients for thrust and friction; it is apparently independent of the size of the wheel and of the width of the blades, but will vary somewhat with the contour of the blades. Prof. Durand's experiments show that the efficiency is not independent of the width of the blade, and it is but fair to assume that the efficiency and other properties of propeller-wheels depend on the size of the wheel and on the material and condition of the surface. It is very desirable, consequently, that the efficiency of large wheels shall be determined from tests on large wheels, and that the factors a and f shall also be determined directly. Unfortunately these properties cannot be determined directly from tests on wheels of ships in service, and the expense and difficulty of experiments on a large scale have prevented such tests as yet.

Thrust and Friction of Propellers.—From a consideration of tests made by Froude on model propellers, Naval Constructor Taylor deduced the following value for the coefficient of friction for propellers:

$$f=0.0160.$$

The thrust coefficient he found to vary with the diameter ratio; he represented a by the following equations:

For four-bladed propellers	....	$a=3.0-0.35d$
" three-bladed "	....	$a=3.3-0.4d$
" two-bladed "	....	$a=3.6-0.5d$

Naval Constructor Taylor says that these factors may be used for manganese-bronze propellers fitted to war-ships and fast passenger-ships.

Thicker and rougher cast-iron blades will have larger coefficients of friction and smaller thrust factors. Some evidence from progressive speed trials shows that the thrust factor is commonly less than given by the above equations even for smooth bronze propellers. Thus the government trials of the U. S. S. *Yorktown* gave 2.8 for a instead of 3.0, which the above equations indicate as the proper value for a three-bladed propeller with a diameter ratio of $10.5 \div 12.5$. Again, trials of the U. S. S. *Manning** gave 2.4 instead of 2.7 for a diameter ratio of $11 \div 12.3$. In each case the blades were wider than the Admiralty blade, and Durand's experiments indicate that the thrust coefficient is less for a wider blade. The investigation of the thrust factor from results of speed trials involves various uncertainties which can be best stated in connection with the discussion of such trials.

It should be noted that thin smooth blades of bronze give a large thrust factor and higher efficiency than rough cast-iron blades, and require more power to turn them; consequently the pitch should be reduced when iron blades are replaced by bronze.

Graphical Integration for A, B, and C.—To obtain values for the functions A, B, and C for a given form of propeller-blade the following method may be used:

Draw an expanded contour of the blade similar to Fig. 199. Find the diameter ratio,

$$d_a = \frac{D_a}{P} = \frac{2r}{P},$$

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for a strip Δr at the distance r from the axis, and take from the table on page 458 the functions α, β, and γ corresponding. Measure the width l of the strip and find the width ratio, $l \div D$; multiply each function by this width ratio, as indicated by equations (32), (33), and (34). Draw a diagram, Fig. 200, with the ratio $r \div D$ for abscissa and the products of the functions by the width ratio for ordinates; after a sufficient number of points are located, curves may be drawn as indicated. The curves will cross the axis of the abscissæ at the

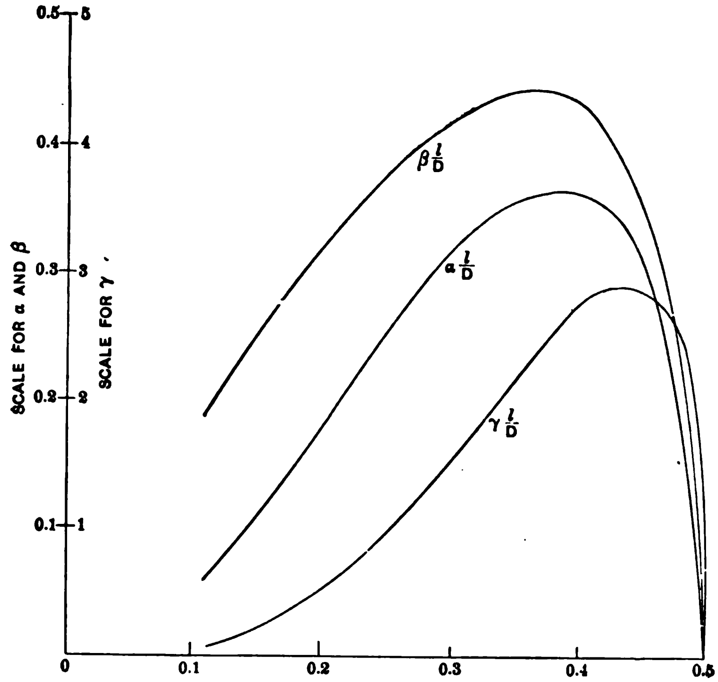

FIG. 200.

point 0.5, which corresponds to the ratio $r \div D$ at the tip of a blade. The products

$$\alpha\frac{l}{D} \quad \text{and} \quad \beta\frac{l}{D}$$

may be conveniently laid off with the same scale as the abscissæ, but the product

$$\gamma\frac{l}{D}$$

FUNCTIONS OF CHARACTERISTICS FOR ADMIRALTY BLADE.

A

Diameter Ratio, $D+P$.	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.40	0.129	0.134	0.140	0.145	0.151	0.156	0.161	0.167	0.172	0.178
0.50	0.184	0.189	0.195	0.201	0.206	0.212	0.218	0.224	0.230	0.236
0.60	0.242	0.248	0.254	0.260	0.266	0.272	0.278	0.284	0.290	0.296
0.70	0.302	0.308	0.315	0.321	0.327	0.334	0.340	0.347	0.353	0.360
0.80	0.366	0.372	0.378	0.385	0.391	0.398	0.404	0.411	0.417	0.424
0.90	0.430	0.437	0.443	0.449	0.456	0.462	0.469	0.475	0.482	0.488
1.00	0.495	0.502	0.508	0.514	0.521	0.527	0.534	0.540	0.547	0.553
1.10	0.560	0.566	0.573	0.579	0.586	0.592	0.598	0.605	0.611	0.618
1.20	0.624	0.631	0.638	0.644	0.650	0.657				

B

0.40	0.416	0.419	0.422	0.425	0.429	0.433	0.437	0.441	0.445	0.449
0.50	0.453	0.457	0.461	0.465	0.469	0.473	0.477	0.482	0.486	0.491
0.60	0.495	0.500	0.504	0.508	0.513	0.517	0.522	0.526	0.531	0.536
0.70	0.540	0.545	0.550	0.554	0.559	0.564	0.569	0.574	0.579	0.583
0.80	0.588	0.593	0.598	0.603	0.608	0.613	0.618	0.623	0.628	0.633
0.90	0.638	0.643	0.648	0.653	0.658	0.663	0.668	0.673	0.678	0.684
1.00	0.689	0.694	0.700	0.705	0.710	0.715	0.721	0.726	0.731	0.737
1.10	0.742	0.748	0.753	0.759	0.764	0.770	0.775	0.781	0.787	0.792
1.20	0.797	0.803	0.809	0.814	0.820	0.825				

C

0.40	0.263	0.279	0.296	0.315	0.333	0.352	0.373	0.393	0.413	0.433
0.50	0.456	0.480	0.504	0.530	0.557	0.583	0.610	0.637	0.666	0.696
0.60	0.728	0.758	0.791	0.825	0.859	0.896	0.932	0.969	1.005	1.044
0.70	1.082	1.125	1.166	1.209	1.252	1.300	1.354	1.406	1.457	1.509
0.80	1.562	1.616	1.671	1.729	1.791	1.853	1.915	1.978	2.041	2.103
0.90	2.170	2.241	2.312	2.383	2.456	2.525	2.604	2.683	2.762	2.843
1.00	2.927	3.010	3.098	3.187	3.274	3.364	3.458	3.556	3.654	3.750
1.10	3.854	3.958	4.061	4.170	4.280	4.393	4.508	4.622	4.739	4.862
1.20	4.984	5.114	5.249	5.385	5.520	5.655				

may have a scale one-tenth as large. The areas under the curves involving α and β are the values of the functions A and B; the area under the curve involving γ is to be multiplied by 10 to find the value of C. If the areas are computed by Simpson's rule or by a similar process, the diagram may be any convenient size; if the area is to be measured by a planimeter, the unit of length may be conveniently taken as 10 inches and then the area in square

FUNCTIONS OF CHARACTERISTICS FOR TAYLOR'S BLADE.

A

Diameter Ratio, $D \div P$.	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.40	0.153	0.158	0.165	0.170	0.176	0.182	0.187	0.194	0.200	0.206
0.50	0.212	0.219	0.225	0.231	0.238	0.245	0.251	0.257	0.265	0.271
0.60	0.278	0.284	0.291	0.298	0.304	0.311	0.318	0.325	0.331	0.338
0.70	0.345	0.352	0.358	0.375	0.383	0.390	0.397	0.404	0.411	0.418
0.80	0.425	0.432	0.439	0.446	0.453	0.460	0.467	0.474	0.481	0.488
0.90	0.495	0.501	0.510	0.516	0.523	0.530	0.537	0.544	0.551	0.558
1.00	0.565	0.572	0.579	0.586	0.593	0.600	0.607	0.614	0.621	0.628
1.10	0.635	0.642	0.649	0.656	0.663	0.670	0.677	0.684	0.691	0.698
1.20	0.705	0.712	0.719	0.726	0.733	0.740				

B

Diameter Ratio, $D \div P$.	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.40	0.472	0.474	0.477	0.480	0.483	0.486	0.489	0.492	0.495	0.498
0.50	0.501	0.504	0.508	0.511	0.515	0.518	0.522	0.527	0.530	0.534
0.60	0.538	0.543	0.547	0.552	0.556	0.561	0.565	0.571	0.576	0.581
0.70	0.587	0.592	0.598	0.603	0.608	0.614	0.620	0.626	0.632	0.637
0.80	0.643	0.649	0.654	0.660	0.666	0.671	0.677	0.683	0.689	0.695
0.90	0.700	0.706	0.712	0.717	0.725	0.730	0.735	0.741	0.747	0.753
1.00	0.759	0.765	0.771	0.777	0.783	0.789	0.795	0.801	0.807	0.813
1.10	0.819	0.825	0.831	0.837	0.843	0.849	0.855	0.861	0.867	0.873
1.20	0.879	0.885	0.891	0.897	0.903	0.909				

C

Diameter Ratio, $D \div P$.	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.40	0.432	0.450	0.459	0.468	0.486	0.495	0.513	0.531	0.549	0.567
0.50	0.585	0.612	0.630	0.657	0.675	0.702	0.729	0.756	0.783	0.819
0.60	0.846	0.882	0.909	0.945	0.990	1.020	1.070	1.120	1.160	1.210
0.70	1.260	1.310	1.360	1.410	1.470	1.520	1.580	1.650	1.710	1.770
0.80	1.840	1.900	1.970	2.040	2.110	2.190	2.260	2.340	2.410	2.490
0.90	2.580	2.670	2.750	2.830	2.920	3.020	3.110	3.210	3.310	3.400
1.00	3.580	3.700	3.820	3.940	4.070	4.180	4.300	4.420	4.550	4.660
1.10	4.810	4.950	5.100	5.250	5.400	5.550	5.700	5.850	5.990	6.130
1.20	6.270	6.420	6.580	6.750	6.920	7.080				

inches is to be divided by 100 to get the values of A and B, and by 10 to get the value of C; or the unit of length may be 20 inches, and the divisors will in that case be 400 and 40.

On pages 464 to 466 will be found tables of the functions A, B, and C for the Admiralty blade, for Taylor's blade, and for the straight-edged blade, all determined for a maximum width ratio of unity. In using these tables the values of A, B, and C are

FUNCTIONS OF CHARACTERISTICS FOR STRAIGHT-EDGED BLADES.

A

Diameter Ratio, $D \div P$.	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.40	0.144	0.150	0.156	0.163	0.170	0.176	0.183	0.189	0.195	0.202
0.50	0.208	0.215	0.221	0.228	0.235	0.241	0.248	0.254	0.261	0.267
0.60	0.274	0.281	0.287	0.294	0.301	0.307	0.314	0.321	0.328	0.334
0.70	0.341	0.348	0.354	0.361	0.367	0.374	0.381	0.388	0.395	0.401
0.80	0.408	0.415	0.422	0.428	0.435	0.442	0.449	0.455	0.462	0.469
0.90	0.476	0.483	0.489	0.496	0.503	0.510	0.517	0.524	0.531	0.537
1.00	0.544	0.551	0.558	0.565	0.572	0.579	0.586	0.593	0.600	0.607
1.10	0.614	0.620	0.627	0.635	0.642	0.649	0.656	0.663	0.670	0.677
1.20	0.684									

B

0.40	0.410	0.414	0.418	0.423	0.427	0.432	0.437	0.442	0.446	0.450
0.50	0.455	0.460	0.465	0.470	0.474	0.479	0.484	0.489	0.494	0.498
0.60	0.503	0.508	0.513	0.518	0.523	0.528	0.533	0.538	0.543	0.548
0.70	0.553	0.558	0.564	0.569	0.575	0.580	0.585	0.591	0.597	0.603
0.80	0.607	0.613	0.618	0.624	0.630	0.635	0.641	0.646	0.652	0.657
0.90	0.663	0.669	0.675	0.680	0.685	0.691	0.697	0.702	0.708	0.714
1.00	0.720	0.725	0.731	0.737	0.743	0.749	0.755	0.761	0.767	0.773
1.10	0.779	0.785	0.791	0.797	0.803	0.810	0.816	0.823	0.829	0.835
1.20	0.840									

C

0.40	0.313	0.331	0.350	0.372	0.395	0.420	0.445	0.470	0.497	0.524
0.50	0.552	0.580	0.610	0.643	0.677	0.712	0.747	0.781	0.815	0.850
0.60	0.887	0.925	0.965	1.005	1.050	1.095	1.142	1.188	1.235	1.283
0.70	1.332	1.383	1.435	1.490	1.545	1.600	1.655	1.715	1.775	1.835
0.80	1.900	1.965	2.040	2.110	2.180	2.250	2.330	2.410	2.490	2.570
0.90	2.650	2.735	2.825	2.915	3.005	3.100	3.195	3.290	3.395	3.495
1.00	3.600	3.705	3.810	3.925	4.040	4.155	4.275	4.400	4.525	4.650
1.10	4.775	4.910	5.045	5.180	5.320	5.465	5.610	5.755	5.905	6.060
1.20	6.225									

to be taken for the diameter ratio $D \div P$ and are to be multiplied by the maximum width ratio $l \div D$. Here D is the diameter of the wheel measured from tip to tip of the blades, P is the pitch, and l is the maximum width of the blade, measured at the middle for the Admiralty blade and for Taylor's blade, and at or near the tip for the straight-edged blade.

Interaction of Propeller and Ship.—Thus far the propeller has been considered to act on undisturbed water, as the model does when carried on a frame in the towing-tank. When a propeller is placed behind a ship it acts on water which is disturbed by the ship, and on the other hand it disturbs the natural flow of water which closes in after the ship. This leads to the consideration of the wake and to what is known as the thrust deduction.

The Wake.—A ship propelled by sails or towed in undisturbed water sets in motion a stream of water in the same direction; this stream or wake may be attributed mainly, if not wholly, to the friction of the water on the skin of the ship. But near the stern of the ship there are other actions that may make the water move in the same direction and influence the wake at that place, namely, the stream-line flow and the effect of the transverse wave. The water behind the stern of a body moving in a frictionless fluid will have a greater pressure and a less velocity relative to the body than water at a distance; this is indicated in an exaggerated way by Fig. 166, which shows stream-lines about an oval. This effect is known as the stream-line wake; it is probable that it has but little influence on a ship with a fine run.

Again, the water in which the propeller is working is affected by the transverse waves generated by the ship. If there should happen to be a crest over the propeller, then the water at that place has a forward motion which increases the velocity of the wake; on the contrary, if there should be a hollow over the propeller, the water will move backward and will decrease the velocity of the wake. The effect of this action can be exhibited by an example, for which purpose we may assume that a ship at the speed of 19 knots has a transverse wave one-fortieth as high as it is long. By the table on page 261 the length of the wave will be 200 feet and its height is consequently 5 feet. The velocity of a particle in its orbit will be

$$\frac{2\pi r}{T} = \frac{\pi d}{L}c = \frac{\pi}{40}c = 1.5 \text{ knots per hour,}$$

where T is the time of the wave in seconds, L is the length in feet,

and c is the speed in knots per hour. Whatever advantage may appear at first sight to come from the presence of a transverse crest over a propeller is more than offset by the addition to resistance from wave interference.

All the three elements that can give velocity to the wake in

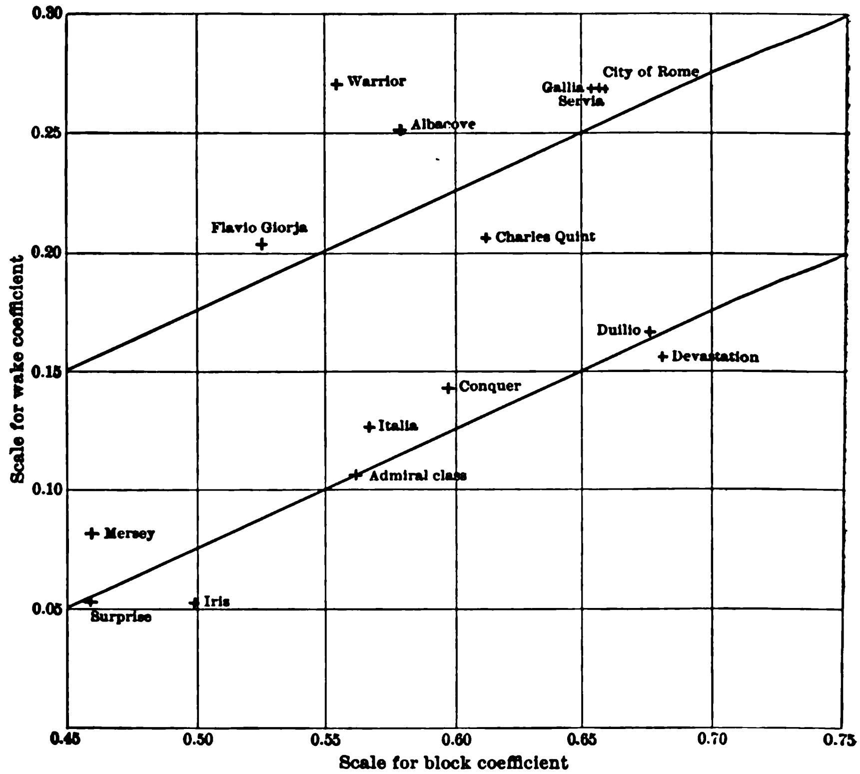

FIG. 201.

which the screw works, namely, friction, stream-lines, and waves, tend to give a varying velocity to the wake. The wake clearly will have a higher velocity near the surface and near the axis of the ship. In addition the wake is likely to be affected by irregular disturbances or eddies. Mr. Froude, from tests on model screws behind models in the towing-tank, concluded that the effect of a turbulent wake on a screw acting in it is substantially the same as

that of a uniform wake having the same mean velocity. He found that the efficiency of a screw in such a wake was slightly greater than when working in undisturbed water, but the gain was not greater than the unavoidable errors of the experiments, and he consequently advised that the same efficiency be attributed to a screw behind a ship as was found from tests in the tank.

The mean value attributed to the wake of a large well-formed ship by Froude is 10 per cent of the speed of the ship. The wake factor is commonly given as the ratio of the velocity of the wake to that of the ship, and may be represented by the letter w; Froude's mean value for w is then 0.1.

Fig. 201 gives the wake for a number of ships, as determined by Froude, plated on the block coefficients of the ships as abscissæ. In this diagram all the ships having more than 20 per cent wake are single-screw ships, and all having less wake are twin-screw ships; there does not appear to be any reason for the distinction shown by the diagram between the two classes of ships. Information from more recent ships is much to be desired. The lines drawn on the diagram have the following equations, which may be used in applying the information represented by it:

Twin-screw ships,

$$w = 0.10 + \tfrac{1}{2}(\text{block coefficient} - 0.55).$$

Single-screw ships,

$$w = 0.20 + \tfrac{1}{2}(\text{block coefficient} - 0.55).$$

Real and Apparent Slip.—The slip of the screw defined on page 449 and used in deducing the work and efficiency of the propeller is

$$s = \frac{\frac{1}{60}PR - v}{\frac{1}{60}PR}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v is the velocity of the car, in feet per second, over the towing-tank.

If it is desired to use the velocity of the car in knots per hour, we shall have for $V =$ knots per hour

$$s_1 = \frac{\frac{1}{60}PR - V\frac{6080}{60 \times 60}}{\frac{1}{60}PR} = \frac{60PR - 6080V}{60PR}. \quad . \quad . \quad . \quad (2)$$

This is known as the apparent slip when dealing with a ship which has a propeller working in the wake astern, and will be affected by a subscript to distinguish it from the real wake. Clearly the speed of the ship in feet per second is

$$v = (1 - s_1)\tfrac{1}{60}PR. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Suppose now that the water in the towing-tank could have imparted to it a velocity equal to that of the wake of a ship, then the model propeller working in the tank (not behind a model of the ship) would be substantially in the same condition as the actual propeller in the wake of the ship. The velocity of the screw through the water is no longer that of the car on the track, but is equal to that velocity multiplied by $(1-w)$, where w is the wake factor. The velocity of the screw through the water will be

$$(1-w)(1-s_1)\tfrac{1}{60}PR; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

comparing this with the velocity which would result if the screw worked in a solid nut or rack, the real slip is

$$s = \frac{\frac{1}{60}PR - (1-w)(1-s_1)\frac{1}{60}PR}{\frac{1}{60}PR}, \quad . \quad . \quad . \quad . \quad (5)$$

whence

$$s = 1 - (1-w)(1-s_1) = w + s_1(1-w). \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \ (1-s) = (1-w)(1-s_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$w = \frac{s - s_1}{1 - s_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The velocity of the screw through the water in feet per second can now be written

$$(1-s)\tfrac{1}{60}PR, \quad \ldots \ldots \quad (9)$$

instead of the form used in (4).

Wake Gain.—Turning to the elements of the theory of the propeller, page 453, Fig. 197, it appears that we have properly, as before,

$$BC=P(1-s), \quad \ldots \ldots \quad (10)$$

where s is the true slip, and the expressions for thrust and turning force, ΔT and ΔV, will remain unchanged.

Coming to the useful and the gross works, ΔW_t and ΔW_u, it appears that the latter is unchanged, but the former should now be obtained by multiplying ΔT by the distance the car (or ship) moves in one second, which is

$$\tfrac{1}{60}PR(1-s_1), \quad \ldots \ldots \quad (11)$$

as above in equation (3). But the factor $(1-s)$ goes through the entire theory of the propeller as a constant; therefore we may calculate the work imparted to the car (or ship) by replacing $(1-s)$ in equations (35) and (37), page 460, by $(1-s_1)$; giving

$$W_t'=\frac{1}{60^3}P^2R^3D^2N(1-s_1)(asA-fB), \quad \ldots \quad (12)$$

$$\text{T.H.P.}=\frac{1}{550\times 60^3}P^2R^3D^2N(1-s_1)(asA-fB), \quad \ldots \quad (13)$$

where the horse-power of equation (13) is that delivered to the thrust-block of the ship and is applied to driving the ship.

If the E.H.P. and T.H.P. are compared, it appears that their ratio is

$$\frac{\text{T.H.P.}}{\text{E.H.P.}}=\frac{1-s_1}{1-s}=\frac{1}{1-w}. \quad \ldots \ldots \quad (14)$$

The advantage of placing the screw propeller in the wake is called the wake gain. It can be represented by the factor

$$\frac{1}{1-w}, \quad \ldots \ldots \quad (15)$$

by which the effective horse-power is to be multiplied to find the thrust horse-power. If the wake factor be made 0.1, as recommended by Froude, then

$$\frac{1}{1-w}=1.11,$$

or the gain is 11 per cent of the effective horse-power.

The wake gain is really that part of the work which is continually expended by the ship to maintain the wake, which can be abstracted from the wake by the propeller. Since the propeller can get only a part of the work expended on the wake, it is evident that the wake should be as small as possible. In general more than half of the resistance of the ship is to be attributed to skin friction, and of this only a small fraction can be recovered by the screw.

Thrust Deduction.—If the screw propeller could be placed a considerable distance behind the ship, it might get the advantage of working in the wake without disturbing the stream-lines about the ship; but it is necessary to place the propellers well under the stern for various reasons; consequently the screw disturbs the stream-lines and reduces the pressure at the stern of the ship. Another way of looking at this matter is to consider that the flow of water toward the propeller is due to a reduction of pressure at the stern which acts as a drag and increases the power required to propel the ship. It is customary to represent the increased power required to overcome this action by a factor

$$\frac{1}{1-t}, \quad \ldots \ldots \ldots \quad (16)$$

where t is called the thrust deduction. Conversely, if a certain power is available for propulsion, the thrust deduction has the effect of reducing the net power (that can be applied to overcome skin and wave-making resistance) by the factor

$$\frac{1-t}{1}. \quad \ldots \ldots \ldots \quad (17)$$

The expression

$$\frac{1-t}{1-w} \quad \ldots \ldots \ldots \quad (18)$$

is sometimes called the hull efficiency; the only certain way of determining this factor is by means of experiments in a towing-tank. Its value is never very far from unity for large well-formed ships, and that value will be attributed to it in our future work. Consequently we may lay aside all consideration of both wake gain and thrust deduction for well-formed ships and may use equations (37) and (39), page 461, to calculate the effective power and efficiency of a propeller, remembering that s is the real slip, which, of course, depends on the wake. The slip may be calculated directly by equation (5), page 470, or it may be calculated from the apparent slip by aid of equation (6); the second method is, of course, equivalent to the first, as it requires the calculation of the apparent slip by equation (1), page 469.

The real slip is commonly calculated roughly by adding together the wake factor and the apparent slip. Thus, if the apparent slip is 10 per cent, and the wake factor is 0.1, the real slip is called 0.20; equation (6) gives

$$s=w+s_1(1-w)=0.1+0.1(1-0.1)=0.19;$$

but the wake is seldom known within one per cent, and an error of one per cent in the real slip has not an important effect on our calculations.

Twin Screws.—A single screw on a shaft protruding through the stern-post (when it is sufficient and convenient) is on the whole the most satisfactory means of propulsion. It works where the wake is strongest, and it is well guarded by the external stern, where that protrudes as on merchant-ships; it requires only one engine; and, finally, it probably has a higher efficiency than twin or triple screws.

The diameter of the screw is usually limited by the draught of the ship; the only exceptions being the screws of torpedo-boats and speed-launches, which boats may have shallow hulls for sake of reducing their resistance. If the power applied to a ship cannot be readily developed on one screw, then two may be used, giving twin-screw propellers. Such ships have the advantage that there are two engines, one of which can propel the ship at about two-thirds her

normal speed; they have the further advantage that the screws can be used to manœuvre or to help manœuvre the ship. These advantages lead to the use of twin screws on all war-ships except small torpedo-boats and gunboats. There is also the further advantage that each engine is smaller and especially requires less head-room than a single engine; this is an advantage on all ships of very large size and power, and more especially for war-ships, which must have the engines under armored decks.

The shaft for a twin-screw propeller must have considerable length outside of the hull, and may be supported by brackets, which again require additional internal framing. This adds weight to the hull of the ship and increases the resistance by an appreciable amount. On the other hand twin screws are commonly more deeply immersed, they may have a more favorable diameter ratio, and they are commonly given three blades, while single screws usually have four blades. On the whole the efficiency of twin-screw propulsion is very nearly as high as that by single screws; for high-speed ships they may be even more efficient, and commonly are required.

Large ocean passenger-ships are so long in proportion to their beam that they may have a fairly large block coefficient and yet have fair easy lines. The external shafts for twin screws are not so long in proportion as for war-ships. Some such passenger-ships avoid external shafts and brackets by the following device: the shafts for the two propellers are brought near together so that the screws as seen in end projection overlap; one screw is then placed a little ahead of the other so that they shall not interfere; an aperture is cut in the deadwood of the stern to allow of the passage of the tips of the blades; the shafts protrude through a "spectacle frame" worked into the frame of the ship just ahead of the aperture; the skin of the ship is splayed out over the shafts as far as the spectacle frame. To avoid too much interference with the flow of water up to the propeller, the fins formed by this construction droop down from the general form of the ship toward the shafts.

Bow-screws.—Screws are properly placed at the stern, for there they get the wake-gain to offset the thrust deduction. A bow-screw throws a stream of water against the bow of the ship and produces

additional pressure which increases the resistance; this is substantially the same in effect as the thrust deduction; there is, of course, no wake gain. The only ships which have bow-screws are double-ended ferryboats and ice-breaking boats; the former are, of course, alike at both ends; the latter have a small deeply immersed screw which draws water away from under the ice and aids the bow in breaking a way through, but this screw has but little propulsive effect.

It has been found that screws which have a large rake of the blades aft are very inefficient when backing, and that, conversely, screws of this form are proper for double-ended ferryboats because the screw which acts as a bow-screw has less deleterious effect than the usual type.

Triple Screws have been used on a few war-ships and on very shallow river boats. The U. S. S. *Columbia* was given three in part because there was then some difficulty in getting very large forgings in the United States, and partly to enable the ship to cruise with one screw only, the wing screws being allowed to revolve freely. Certain European war-ships have been given a splinter-deck at the lower edge of the armor-deck as well as an armored deck at the upper edge; this has led to the use of three screws to get smaller and lower engines. Engineer-in-chief Melville, U. S. N., has proposed the use of three screws of unequal size, the middle screw being driven by an engine having double the power of the two wing engines, and to be used alone except when full speed is demanded.

Inclined Propeller-shafts.—There is some advantage, especially for shallow-draught ships, in inclining the propeller-shaft downwards from the engine to the screw. In much the same way twin-screw shafts are often inclined outwardly to spread the screws so that they shall not come too near the hull. Both of these devices are to be avoided if possible, as they are liable to make the screws run irregularly, and spreading twin-screw shafts interferes with the manœuvring power of the ship.

It has already been pointed out that the wake of a ship is stronger near the surface and near the axis of the ship. A single screw consequently works with a greater slip at the top than at the bottom; to get the same slip at both places the shaft should be inclined upwards

from the engine; inclining the shaft downwards of course increases the irregularity. Now the thrust of a propeller (or of an element of the blade) increases with the slip, and consequently each blade experiences a greater resistance near the top of its revolution and a less resistance near the bottom, and this tends to set up a vibration which is felt most unpleasantly near the stern. There are, however, other causes which lead to the same result, and we cannot expect to avoid pulsations by sloping the propeller-shaft upwards even if the construction of the ship should allow.

It is evident that the shafts of twin screws should be inclined inwards and upwards if they are to work as in a uniform wake, but if inclined at all, the inclination is always downwards and outwards in practice. The only exception is that a certain cable-ship had the shafts of its twin screws inclined inwards to aid in manœuvring. The turning moment of a propeller-shaft tending to swing the ship from a straight course may be obtained by multiplying its thrust by the perpendicular let fall on it from the centre of lateral resistance of the ship; this for our present purpose can be assumed to be at the middle of its length. Since the engines are always abaft the middle of the ship, spreading twin-screw shafts will shorten the arm of this turning moment. Some twin-screw ships with spreading propeller-shafts cannot be turned by their propellers alone even though one is driven forward and the other backward.

Engine Efficiency and Coefficient of Propulsion.—A marine engine may be expected to lose from 10 to 15 per cent of its power in friction, variously distributed at the pistons, crank-pins, main bearings, thrust-block, and elsewhere; the power required to drive the air-pumps from the main engines is variously estimated at 3 to 7 per cent of the power of the engine. The mechanical efficiency of the engine may consequently be estimated at from 0.8 to 0.9. The efficiency of the propeller if well designed and well made will vary from 0.50 to 0.70. The ratio between the effective horse-power and the indicated horse-power (neglecting the hull efficiency) can be obtained by multiplying the mechanical efficiency of the engine by the efficiency of the propeller; it will vary from

$$0.8 \times 0.5 = 0.4 \quad \text{to} \quad 0.9 \times 0.70 = 0.6.$$

This factor is commonly called the coefficient of propulsion, and is assumed to vary from

$$0.5 \text{ to } 0.55$$

for well-designed ships.

Taylor's Method for Propellers.—As already stated at the beginning of the theoretical investigation of the action of the propeller, the method given here is, with little variation, that developed by Naval Constructor Taylor, who has given it a form that can be conveniently used in practice. In particular he proposed the use of the functions α, β, and γ and A, B, and C, and computed tables for the form of blade proposed by him.

The method of designing a propeller by aid of the functions A, B, and C and the equations in which they occur is conveniently shown by means of an example.

Let it be required to find the dimensions of a propeller for a ship which is driven at 16 knots per hour by an engine which develops 3000 horse-power, the wake factor being 0.1. The propeller will have four blades, and the engine efficiency will be assumed to be 0.85.

Assume a real slip of 0.2, which will give for the apparent slip

$$s_1 = 1 - \frac{1-s}{1-w} = 1 - \frac{1-0.2}{1-0.1} = 0.1 \text{ nearly.}$$

Let us try a diameter ratio $D \div P = 1$; for this ratio the equation on page 462 gives

$$a = 3.0 - 0.35 = 2.65,$$

and table of characteristics for the Admiralty blade gives

$$\begin{aligned} &\text{Function for } A = 0.495; \\ &\quad\text{''}\qquad\text{''}\quad B = 0.689; \\ &\quad\text{''}\qquad\text{''}\quad C = 2.93; \end{aligned}$$

so that for the standard Admiralty blade 0.2 of the diameter wide

$$A = 0.0990, \qquad B = 0.1378, \qquad C = .586.$$

The coefficient of friction may be given the standard value $f=0.016$.

By equation (39) the efficiency is

$$e=(1-s)\frac{asA-fB}{asA+fC}.$$

$$\therefore\ e=.8\ \frac{2.65\times 0.2\times 0.099-0.016\times 0.1378}{2.65\times 0.2\times 0.099+0.016\times 0.586};$$

$$\therefore\ e=0.65.$$

This propeller efficiency, with 0.85 for the engine efficiency, makes the efficiency of propulsion

$$0.65\times 0.85=0.55,$$

so that the effective horse-power is

$$3000\times 0.55=1650.$$

Equation (37), page 461, gives for the effective horse-power

$$\text{E.H.P.}=\frac{1}{550\times 60^3}P^3R^3D^2N(1-s)(asA-fB),$$

in which there is known for this problem all but the pitch, diameter, and revolutions of the propeller; in addition we know that the speed of the ship in feet per minute is equal to the product of the pitch by the revolutions per minute multiplied by one minus the apparent slip, so that

$$16\times 6080\div 60=.9PR.$$

$$\therefore\ PR=1800.$$

Inserting the known values in the above equation, we have

$$1650=\frac{(1800)^3}{550\times 60^3}D^2\times 4\times .8(2.65\times .2\times 0.099-0.016\times .1378),$$

$$D^2=\frac{1650\times 550\times 60^3}{(1800)^3 4\times .8(2.65\times 0.2\times 0.099-0.016\times 0.1378)},$$

$$D=14.4\text{ feet.}$$

The diameter ratio being 1, the pitch is also 14.4 feet, and the engine makes

$$R = 1800 \div 14.4 = 125$$

revolutions per minute.

Should it be found that the size of the propeller-wheel or the number of revolutions is unsuited to the ship for which it is intended, it will be necessary to choose a different diameter ratio and solve again, bearing in mind that a large diameter ratio gives a small wheel and a large number of revolutions. It will be well to proceed systematically, using several diameter ratios and thus getting a regular series of results from which the best wheel for the purpose can be selected. For example, the problem just discussed, solved for four diameter ratios, gives the following table:

STANDARD SLIP AND WIDTH.

Diameter Ratio.	Efficiency.	Diameter, Feet.	Pitch, Feet.	Revolutions per Minute.
0.5	70	23.6	47.0	39
0.7	69	18.0	25.7	70
0.9	66	15.3	17.0	106
1.0	65	14.4	14.4	125

This table is purposely made with a wide range of diameter ratios to exhibit the effect of changing that function; in practice three diameter ratios, advancing by tenths or even smaller intervals, will be found sufficient, provided the designer can originally select the proper diameter ratio with good degree of certainty. After a series of results have been obtained the best solution of the problem can be had by interpolation either numerically or graphically.

Thus far the problem has been solved using the standard real slip of 0.2 and the standard width of blade equal to 0.2 of the diameter; this is the preferable method of procedure where it can be applied, but in some cases it will give either too large a wheel or too high a speed of revolution for the engine. If a small wheel and moderate speed of engine are desired, we may resort to one of two expedients: (1) increase the slip, (2) increase the width of the blade; the second method is the more efficacious, but should be resorted to

only when imperative, and the width should seldom, if ever, be made greater than 0.3 of the diameter. To show the effect of increasing the slip, the following table has been computed with 0.25 for the slip:

STANDARD WIDTH, SLIP 0.25.

Diameter Ratio.	Efficiency.	Diameter, Feet.	Pitch, Feet.	Revolutions per Minute.
0.5	0.67	19.2	38.4	51.0
0.7	0.66	14.9	21.3	91.6
0.9	0.65	12.5	13.9	140.5
1.0	0.63	11.6	11.6	168.0

The effect of increasing the slip for a given diameter ratio is to decrease the diameter and to increase the number of revolutions; but a comparison of this table with the table for 0.20 slip will show that by choosing a different diameter ratio a smaller wheel for a given number of revolutions can be obtained. Thus, interpolation in the last table gives 125 revolutions and a diameter of 13.1 feet for a diameter ratio of 0.84; these results are approximate only because the intervals of the table are too large for direct interpolation, but they exhibit the effect of a change in the slip.

Although the theory now in use does not provide for a change of efficiency when the width of the blade is increased, it is probable that in practice there will be some loss of efficiency, especially for a slip of more than 0.2, as will appear from a reference to Fig. 196, page 452. To show the effect of changing the width of the blade. the following table has been computed for a blade having the maximum width, $0.3D$:

STANDARD SLIP, WIDTH $0.3D$.

Diameter Ratio.	Efficiency.	Diameter, Feet.	Pitch Feet.	Revolutions per Minute.
0.5	70	19.3	38.0	47.0
0.7	69	14.7	21.0	86.7
0.9	66	12.5	14.0	129.0
1.0	65	11.6	11.6	152.0

This table can be readily obtained from the table for standard width of blade on the preceding page because the characteristics

A and B are proportional to the width of the blade; therefore in the equation for D^2 on page 478 these functions become

$$A' = \tfrac{3}{2} \times 0.0990 \quad \text{and} \quad B' = \tfrac{3}{2} \times 0.1378,$$

and consequently the diameter of the wheel becomes $\sqrt{\tfrac{2}{3}}$ as large as for the standard width. The number of revolutions is inversely proportional to the diameter and increases in the ratio $1 : \sqrt{\tfrac{3}{2}}$. Interpolation in this table shows that a speed of revolution of 125 turns per minute will give a wheel 12.7 feet in diameter; this illustrates the fact that increasing the width of blade is an efficacious way of reducing the diameter of the wheel. It should be remembered, however, that the thrust increases less rapidly than the width of the blade, especially for widths greater than 0.2 of the diameter. The logical method would be to make an allowance for this effect by decreasing the thrust-factor a, but unfortunately we have not the experimental work for this method. Until some better way is found, it may be suggested that only a portion of the added width be allowed for; thus if the width be made 0.3 of the diameter, we may make our calculations as though the width were 0.26 or 0.28 of the diameter.

It may be well to call attention here to the fact that progressive speed-trials of boats with wide blades show comparatively small values for the thrust coefficient. This confirms the opinions given above, but unfortunately the determination of the thrust coefficient from those trials involves the assignment of a value to the wake coefficient by a method that is not entirely satisfactory, and further the whole number of trials available is too small to draw definite conclusions. The matter will be referred to in connection with progressive speed-trials.

Barnaby's Method.—From tests made in the Admiralty towing-tank Mr. R. E. Froude developed a method of designing propellers which has since been recast by Mr. Sidney Barnaby in his book on marine propellers, and is there presented in the form of a table of functions to be used in simple equations for the diameter and revolutions of a propeller.

In the preparation of this table Mr. Barnaby has chosen a standard wake of 0.10, has assumed that the coefficient of propulsion

is 0.50, and has taken a standard width of $0.2D$. It will be shown later how allowances can be made for changes in these characteristics. The table is made for four-bladed screws, but can be used also for three- or two-bladed screws, taking account of Froude's determinations of the relative effects of these several numbers of blades.

The size and speed of revolutions of propellers are expressed by equations as follows:

$$\text{Disc-area} = C_A \times \frac{\text{I.H.P.}}{(\text{speed in knots})^3};$$

$$\text{Revolutions} = C_R \times \frac{\text{speed in knots}}{\text{diameter in feet}};$$

or, using the following notation:

A = area of disc, i.e., of a circle by the tips of the blades;
R = revolutions per minute;
D = diameter in feet;
V = speed of the ship in knots per hour;
p = pitch ratio $= P \div D$,—

$$A = C_A \frac{\text{I.H.P.}}{V^3}, \qquad (1)$$

$$R = C_R \frac{V}{D}, \qquad (2)$$

$$k = \frac{\text{I.H.P.} \times R^2}{V^5}. \qquad (3)$$

The third equation is given as an aid in proportioning a propeller which must have a given speed of revolutions.

The apparent slip of the propeller is

$$s_1 = \frac{PR - V6080 \div 60}{PR}, \qquad (4)$$

or, replacing R by its value from equation (2),

$$s_1 = \frac{PC_R\frac{V}{D} - 101.3V}{PC_R\frac{V}{D}}.$$

$$\therefore s_1 = \frac{pC_R - 101.3}{pC_R}, \qquad (5)$$

from which the slip is readily calculated for a given case.

The functions C_A, C_R, and k are given in the table on pages 484 and 485. This table gives a wide latitude to the designer, who may vary the propeller according to his judgment or the requirements of the case. For large ships column 5 between double lines is likely to give satisfactory proportions. If higher speeds of revolution are desired, columns to the right may be selected; if slower speeds are desired, columns to the left. It is to be noted that columns near the one named have the maximum efficiency, and columns near the extremities have less efficiency.

To show the method of using Barnaby's tables, take the problem on page 477, namely, to find the dimensions of a propeller for a ship which is propelled at 16 knots per hour by an engine which develops 3000 horse-power.

The functions from column 6 and for a pitch ratio 1 are

$$C_A=215, \qquad C_R=112;$$

consequently

$$A=215\times\frac{3000}{16^3},$$

$$A=157.8,$$

$$D=14.2,$$

$$R=112\times\frac{16}{14.2}=126,$$

$$s_1=\frac{pC_R-101.3}{pC_R}=\frac{112-101.3}{112}=0.09.$$

This result is for four blades; had a three-bladed propeller been desired, then from Froude's comparison on page 450 it appears that the thrust of a three-bladed screw will be 0.865 as much as that of a four-bladed screw of the same diameter and proportions. We may consequently use the standard table provided that we take a fictitious

BARNABY'S TABLE FOR PROPELLERS.

Pitch Ratio.	63%			66%			67%			68%			69%			69%			69%		
	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R
0.8	468	25.00	122	371	33.00	125	304	42.40	128	255	52.90	131	215	65.60	134	182	82.10	138	157	101.00	142
0.9	506	18.40	109	403	24.50	112	329	31.10	114	275	39.10	117	234	48.50	120	199	59.80	123	170	74.60	127
1.0	546	14.10	99	433	18.50	101	355	24.00	104	297	29.70	106	251	37.10	109	215	45.70	112	184	56.50	115
1.1	585	11.10	91	465	14.70	93	380	18.60	95	319	23.20	97	270	29.10	100	230	35.50	102	196	44.20	105
1.2	625	8.65	83	495	11.40	85	405	14.60	87	340	18.70	90	288	23.10	92	245	28.90	95	210	35.10	97
1.3	665	7.00	77	528	9.28	79	431	11.90	81	361	14.90	83	306	18.40	85	261	23.30	88	224	29.00	91
1.4	704	5.78	72	559	7.70	74	456	9.95	76	382	12.50	78	325	15.40	80	276	19.10	82	236	24.00	85
1.5	742	4.75	67	590	6.32	69	482	8.20	71	403	10.40	73	342	12.90	75	291	16.10	77	250	19.60	79
1.6	780	4.00	63	620	5.36	65	507	6.98	67	425	8.80	69	360	11.00	71	307	13.70	73	263	16.80	75
1.7	820	3.45	60	651	4.64	62	533	5.85	63	446	7.43	65	378	9.32	67	322	11.60	69	276	14.30	71
1.8	860	2.96	57	682	3.87	58	558	5.08	60	468	6.43	62	396	8.10	64	339	10.10	66	290	12.50	68
1.9	898	2.55	54	713	3.33	55	584	4.36	57	488	5.60	59	415	7.05	61	353	8.85	63	304	10.90	65
2.0	940	2.17	51	745	2.95	53	609	3.91	55	510	4.82	56	432	6.10	58	369	7.68	60	315	9.60	62
2.1	975	1.93	49	775	2.63	51	635	3.35	52	530	4.32	54	450	5.50	56	384	6.64	57	329	8.30	59
2.2	"	"	"	807	2.34	49	660	2.97	50	552	3.85	52	469	4.88	54	400	5.92	55	342	7.45	57
2.3	"	"	"	838	2.07	47	685	2.65	48	573	3.44	50	486	4.36	52	415	5.30	53	355	6.70	55
2.4	"	"	"	869	1.84	45	710	2.44	47	595	3.04	48	505	3.90	50	430	4.74	51	369	5.99	53
2.5	"	"	"	900	1.68	44	736	2.16	45	616	2.81	47	523	3.46	48	445	4.42	50	381	5.56	52
	1			2			3			4			5			6			7		

BARNABY'S TABLE FOR PROPELLERS.—*Continued.*

Pitch Ratio.	69%			68%			67%			66%			64%			63%		
	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R	C_A	k	C_R
0.8	135	122.00	145	115	154.00	150	100	189.00	155	86	234.00	160	75	285.00	165	65	355.00	171
0.9	146	92.50	131	125	115.00	135	109	140.00	139	93	175.00	144	81	215.00	149	71	262.00	154
1.0	157	71.00	119	135	88.10	123	116	109.00	127	100	135.00	131	88	163.00	135	76	202.00	140
1.1	169	55.30	109	144	70.00	113	125	84.50	116	107	106.00	120	93	130.00	124	82	157.00	128
1.2	180	44.50	101	154	55.20	104	132	69.50	108	115	84.10	111	100	104.00	115	87	128.00	119
1.3	191	35.60	93	163	45.20	97	141	55.60	100	122	68.40	103	106	85.00	107	93	104.00	111
1.4	202	29.40	87	173	36.60	90	150	46.20	94	129	57.10	97	112	70.00	100	98	87.00	104
1.5	213	24.80	82	183	31.00	85	158	38.50	88	136	47.80	91	119	58.30	94	104	72.50	98
1.6	225	21.10	78	193	26.00	80	166	32.50	83	144	41.30	87	125	49.70	89	109	62.50	93
1.7	236	17.70	73	202	22.50	76	175	28.00	79	151	35.00	82	131	43.20	85	115	53.00	88
1.8	248	15.50	70	212	19.70	73	182	24.30	75	159	30.00	78	138	37.30	81	120	46.10	84
1.9	259	13.60	67	222	16.80	69	191	21.20	72	166	26.60	75	145	33.00	78	125	41.10	81
2.0	270	11.90	64	231	15.20	67	200	18.70	69	173	23.50	72	150	29.50	75	131	36.00	77
2.1	280	10.40	61	241	13.30	64	208	16.50	66	180	20.80	69	157	26.00	72	136	32.50	75
2.2	291	9.40	59	250	12.00	62	217	14.80	64	187	18.80	67	162	23.00	69	142	28.70	72
2.3	303	8.41	57	260	10.50	59	225	13.40	62	194	16.60	64	169	20.90	67	148	25.20	69
2.4	315	7.54	55	270	9.45	57	232	12.20	60	202	14.90	62	175	18.90	65	153	23.00	67
2.5	326	7.02	54	280	8.81	56	240	11.00	58	209	13.50	60	181	17.20	63	159	21.00	65
		8			9			10			11			12			13	

horse-power obtained by dividing the actual horse-power by 0.865. Using the same functions, we get

$$A=215\times\frac{3000}{0.865\times16^3},$$

$$A=182,$$

$$D=15.2,$$

$$R=112\times\frac{16}{15.2}=118.$$

If the same number of revolutions can be used as for four blades. namely 126, then the diameter will be very nearly 15 feet; this result can be obtained by making a calculation for the pitch ratio 0.9 and interpolating.

In general it will be found convenient to proceed in the manner illustrated, selecting constants for one or more pitch ratios and from one or more columns, and finally choosing that set of dimensions which appear best fitted for the case. Sometimes either the diameter is fixed or the number of revolutions, in which case the following methods may be used:

Suppose that the diameter must not exceed 14 feet; then we have

$$A=\frac{\pi D^2}{4}=\frac{\pi 14^2}{4}=C_A\frac{3000}{16^3},$$

$$C_A=\frac{\pi 14^2\times16^3}{4\times3000}=210.$$

This value for the function is found in column 7 for the pitch ratio 1.2; the corresponding number of revolutions is

$$R=97\times\frac{16}{14}=111,$$

and the slip is

$$s_1=\frac{1.2\times97-101.3}{1.2\times97}=0.13.$$

This direct solution is the most obvious one; another way would be to interpolate between the pitch ratios 0.9 and 1.0 in column 6, or between columns 6 and 7 for the pitch ratio 1.0.

Suppose now the number of revolutions to be fixed at 125 per minute; then we will first calculate the function k by equation (3), getting

$$k=\frac{3000\times \overline{125}^{2}}{\overline{16}^{5}}=44.7;$$

this value comes between pitch ratios 1 and 1.1, column 6, and will lead to nearly the same solution as given at the beginning of this problem, namely, to 14.15 feet diameter.

The derivation of the constant k may be made as follows: Transposing equations (1) and (2) so as to bring all the variables to the right-hand member,

$$A=\frac{\pi D^2}{4}=C_A\frac{\text{I.H.P.}}{V^3},$$

$$\frac{\pi}{4}\frac{1}{C_A}=\frac{\text{I.H.P.}}{D^2V^3},$$

$$C_R^2=\frac{R^2D^2}{V^2};$$

whence, multiplying, we have

$$k=\frac{\pi}{4}\frac{C_R^2}{C_A}=\frac{\text{I.H.P.}\times R^2}{V^5}.$$

The function k can be found for a given problem by equation (3), as already shown, and can be used as a guide to the places in the tables that can be used for solving the problem.

Mr. Barnaby makes no provision for variation in width of the blade, but obviously the method proposed by Naval Constructor Taylor can be used, namely, to make the thrust proportional to the width of the blade; since the speed of the ship is fixed, this is equivalent to making the horse-power proportional to the width of the blade.

Suppose that the blade is made 0.3 as wide as the diameter of

the wheel; then we may solve for two-thirds of the power, as follows:

$$A = C_A \frac{\frac{2}{3} \times \text{I.H.P}}{V^3},$$

$$A = 215 \times \frac{2000}{16^3} = 105.2,$$

$$D = 11.6,$$

$$R = C_R \frac{V}{D} = 112 \frac{16}{11.6} = 155.$$

There remains now one more factor to take into consideration, namely, the wake factor, which is allowed for by the simple expedient of multiplying the speed of the ship by a factor to be selected from the following table:

FACTOR TO ALLOW FOR WAKE.

Wake, w.	Factor.	Wake, w.	Factor.	Wake, w.	Factor.
0.00	1.10	0.11	0.99	0.21	0.88
0.01	1.09	0.12	0.98	0.22	0.87
0.02	1.08	0.13	0.97	0.23	0.86
0.03	1.07	0.14	0.96	0.24	0.84
0.04	1.06	0.15	0.94	0.25	0.83
0.05	1.05	0.16	0.93	0.26	0.82
0.06	1.04	0.17	0.92	0.27	0.81
0.07	1.03	0.18	0.91	0.28	0.80
0.08	1.02	0.19	0.90	0.29	0.79
0.09	1.01	0.20	0.89	0.30	0.78
0.10	1.00				

The proper wake factor for a certain ship may be computed by one of the equations on page 469.

If the wake factor for a given ship is not 0.1 as assumed by Barnaby, but is 0.15, then from the table the speed is to be multiplied by the factor 0.94. Applying this to the problem already discussed,

$$A = 215 \times \frac{3000}{(16 \times 0.94)^3} = 189.6,$$

$$D = 15.6,$$

$$R = 112 \frac{16 \times .94}{15.6} = 108.$$

The justification of this method will appear in a comparison of Taylor's and Barnaby's methods.

Comparison of Taylor's and Barnaby's Methods.—The theory of the propeller derived from Froude's investigations leads to an expression for the effective horse-power given by equation (37), page 461:

$$\text{E.H.P.}=\frac{1}{550\times 60^3}P^3R^3D^2N(1-s)(asA-fB). \quad . \quad . \quad (1)$$

Replacing the disk area by the proper expression in terms of the diameter, and then solving Barnaby's equation for the power,

$$\text{I.H.P.}=\frac{1}{C_A}\frac{\pi D^2}{4}V^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving equation (4), page 482, for V,

$$V=(1-s_1)\frac{60}{6080}PR; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

replacing $(1-s_1)$ by its value in terms of the wake factor and the true slip from equation (7), page 470,

$$V=\frac{(1-s)60}{(1-w)6080}PR. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Mr. Barnaby takes 0.1 for the standard wake factor, so that

$$V=\frac{1}{0.9}(1-s)\frac{60}{6080}PR. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Allowing for the coefficient of propulsion chosen, namely 0.5, we have

$$\text{E.H.P.}=\frac{1}{2C_A}\frac{\pi D^2}{4}\left(\frac{1}{0.9}\times\frac{60}{6080}\right)^3(1-s)^3P^3R^3. \quad . \quad . \quad (6)$$

Equating the two values for effective horse-power, making $N=4$ in equation (1) since Barnaby's tables are for four-bladed propellers,

$$\frac{4}{550\times 60^3}(1-s)(asA-fB)=\frac{1}{C_A}\frac{\pi}{8}\left(\frac{1}{0.9}\times\frac{60}{6080}\right)^3(1-s)^3,$$

$$C_A=\frac{\pi}{32}\frac{550\times 60^6}{(0.9\times 6080)^3}\frac{(1-s)^2}{asA-fB}. \quad . \quad . \quad . \quad . \quad (7)$$

This shows that the function C_A varies with the slip and the pitch ratio or diameter ratio, as indeed is evident from the use of Barnaby's table. It also allows us to calculate values of the function from the general theory of the propeller, or to deduce values of the thrust coefficient a from Barnaby's tables.

For example, the apparent slip as computed for the pitch ratio 1 in column 6 of the table is 0.09; inserting this and the proper values for the functions A and B (namely 0.099 and 0.1378) in equation (7) gives for the function

$$C_A = 196,$$

which in this case is identical with the factor in Barnaby's table. The values of the other constants required for this computation are

$$a = 2.65, \qquad f = 0.016.$$

This and the comparison of problems solved by the two methods show that they give substantially the same result. Taylor's method gives more direct control, but Barnaby's is the more readily applied.

Returning now to Barnaby's correction for wake factors other than $w = 0.1$, it appears that in equation (4), on the preceding page, we have

$$V = \frac{1-s}{1-w} \times \frac{60}{6080} PR,$$

and that $1-w$ is made equal to 0.9, that is, the speed of the ship is multiplied by 0.9 as follows:

$$0.9V = (1-s)\frac{60}{6080}PR. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If any other wake factor is required for a given problem, then we should multiply V by $1-w$ instead of 0.9. This can be done in effect by replacing V by

$$\frac{1-w}{0.9}V,$$

so that we shall have

$$0.9\left(\frac{1-w}{0.9}V\right) = (1-s)\frac{60}{6080}PR. \quad . \quad . \quad . \quad . \quad (9)$$

The factors in the wake-factor diagram are found in this way; thus for a wake of 0.15 the factor by which V is multiplied is

$$\frac{1-0.15}{0.9}=0.94.$$

Barnaby treats this matter in a somewhat different form, but his method is essentially the same.

Direct Comparison.—When all the conditions for a satisfactory ship are known there is an advantage in determining a propeller for a similar ship by direct comparison even though the method be crude.

Referring to equation (37), page 461,

$$\text{E.H.P.}=\frac{1}{550\times 60^3}P^3R^3D^2N(1-s)(asA-fB),$$

it is evident that if the wake factor is known for a given ship and all the particulars of the propeller, there will remain only two undetermined quantities, a and f; and that if a value be assigned to the factor f for friction, a value may be computed for the thrust factor a. This special thrust factor may be used in designing the propeller for a similar ship, and such a thrust factor will give similar results even if it is not absolutely correct.

If the diameter ratio for the propeller of a new ship be taken the same as for a ship already built, and if the wake factors be assumed to be the same, then the above equation may lead to the proportion

$$(\text{I.H.P.})_1 : (\text{I.H.P.})_2 : : V_1{}^3D_1{}^2 : V_2{}^3D_2{}^2, \quad . \quad . \quad . \quad . \quad (1)$$

provided that the coefficient of propulsion is assumed to be the same for the two ships, for we may replace P^3R^3 by V^3 from equation (2), page 470. This proportion can be deduced also from Barnaby's equation for disk area on page 482.

From the theory of mechanical similitude ships that have speeds proportional to the square roots of their lengths have horse-powers proportional to the $\frac{7}{6}$ powers of their displacement, or to the $\frac{7}{2}$ powers of their lengths. This reduces the proportion (1) to

$$L_1{}^{\frac{7}{2}} : L_2{}^{\frac{7}{2}} : : L_1{}^{\frac{3}{2}}D_1{}^2 : L_2{}^{\frac{3}{2}}D_2{}^2,$$

or

$$L_1 : L_2 : : D_1 : D_2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which shows that the theories of the propeller developed by Taylor and Barnaby both conform to the theory of mechanical similitude.

The relation of the speed to the pitch and revolutions already quoted gives

$$V_1 : V_2 : : P_1R_1 : P_2R_2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or, replacing V by $\sqrt{L}$ and P by $D \times p$,

$$\sqrt{L_1} : \sqrt{L_2} : : R_1D_1 : R_2D_2 : : R_1L_1 : R_2L_2,$$

so that

$$R_1 : R_2 : : \frac{1}{\sqrt{L_1}} : \frac{1}{\sqrt{L_2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the revolutions are inversely as the square roots of the lengths.

From proportion (2) it is apparent that

$$D_1^2 : D_2^2 : : (\text{area midship section})_1 : (\text{area midship section})_2,$$

which makes it possible to proportion propellers to the midship section, provided other conditions are conformed to.

The proportions (2) and (4) are useful in the making and discussion of tank experiments on models of ships with propellers in place. Since the skin resistance of models is relatively greater than that of ships, a model cannot be propelled by the corresponding propeller, but the model must be towed and the propeller must be driven independently.

Seaton's Method.—In the Pocket Book for Marine Engineers, by Seaton and Rounthwaite, there are given the following rules for finding the dimensions of propellers, the letters being given the same significance as in Taylor's and Barnaby's methods:

Apparent slip, $$s_1 = \frac{PR - 101.3V}{PR}. \quad . \quad (1)$$

Pitch, $$P = \frac{101.3V}{(1-s)R}. \quad . \quad . \quad (2)$$

Diameter, $$D = K\sqrt{\frac{\text{I.H.P.}}{\left(\frac{P \times R}{100}\right)^3}}. \quad . \quad . \quad (3)$$

Total developed area of blades $= C\sqrt{\dfrac{\text{I.H.P.}}{R}}. \quad . \quad . \quad (4)$

The constants K and C are to be selected from the following table:

SEATON'S TABLE OF PROPELLER COEFFICIENTS.

Description of Vessel.	Approximate Speed in Knots.	No. of Screws.	No. of Blades per Screw.	Values of K.	Values of C.	Usual Material of Blades.
Bluff cargo-boats	8–10	Single	4	17 –17.5	19 –17.5	Cast iron
Cargo-boats —moderate lines	10–13	do.	4	18 –19	17 –15.5	Cast iron
Passenger- and Mail-boats —fine lines	13–17	do.	4	19.5–20.5	15 –13	Cast iron or steel
Passenger- and Mail-boats —fine lines	do.	Twin	4	20.5–21.5	14.5–12.5	Cast iron or steel
Passenger- and Mail-boats —very fine	17–22	Single	4	21 –22	12.5–11	Gun-metal or bronze
Passenger- and Mail-boats —very fine	do.	Twin	3	22 –23	10.5–9	Gun-metal or bronze
Naval Vessels —very fine	16–22	do.	4	21 –22.5	11.5–10.5	Gun-metal or bronze
Naval Vessels —very fine	do.	do.	3	22 –23.5	8.5– 7	Gun-metal or bronze
Torpedo-boats —very fine	20–26	Single	3	24 –27	7– 5.5	Bronze or forged steel

Seaton recommends that the apparent slip be taken at 8 or 10 per cent, but says that 12 to 18 per cent may be allowed.

The equation for diameter is in much the same form as the equations in Taylor's and Barnaby's methods, except that no allowance is made for varying pitch-ratio and slip; the equation for slip (or for pitch) is, of course, identical. The number of revolutions appears to be left to the judgment of the designer except as he is controlled by the conditions of service of the ship.

It is interesting to make a more explicit comparison for a special case. Let the apparent slip be made 0.1, then

$$PR = V \times 101.3 \div 0.9.$$

For single-screw passenger-ships the value of K is 21 to 22. Taking the first, we have, from equation (3),

$$\frac{\pi D^2}{4}=\frac{\pi}{4}21^2\frac{\text{I.H.P.}}{\left(\frac{V\times 101.3\div 0.9}{100}\right)^3}.$$

$$\therefore\ \frac{\pi D^2}{4}=\frac{\pi}{4}21^2\left(\frac{0.9\times 100}{101.3}\right)^3\frac{\text{I.H.P.}}{V^3}=245\frac{\text{I.H.P.}}{V^3}.$$

The numerical coefficient is equal to C in Barnaby's table, column 5, between 0.9 and 1 pitch ratio, and in column 6, at 1.2 pitch ratio.

Area and Thrust.—A very simple and direct though somewhat crude way of dealing with propellers is to compare the area of all the blades with the indicated thrust; for this purpose the developed area is usually taken, though sometimes the area of the projection on a transverse plane is preferred. The indicated thrust is an arbitrary quantity derived from the equation

$$\text{Indicated thrust}=\frac{\text{I.H.P.}\times 33000}{PR}. \quad . \quad . \quad . \quad . \quad (1)$$

The developed area of all of the blades may then be found from the expression

$$\text{Area of blades}=\frac{\text{Indicated thrust}}{K}=\frac{\text{I.H.P.}\times 33000}{KPR},$$

$$K=1000 \text{ to } 1500.$$

If allowance were made for the slip and the coefficient of propulsion, the indicated thrust could be interpreted in terms of mean pressure in pounds per square foot of developed area. It may be noted that

$$K\div 144=7 \text{ to } 11,$$

which will be found instructive when compared with the discussion of cavitation.

Towboat Propellers.—The conditions of service of a towboat are so peculiar that the general methods of design cannot be used with satisfaction. Towboats are commonly short and fairly full on the water-lines, but have a good rise of floor, so that the wake coefficient is not unduly large. Running free, the slip is not exces-

sive, and it is probable that the hull efficiency is not far from unity. When the boat is towing the speed will be from 5 to 6 knots an hour, which is somewhat less than half the speed when running free. The apparent slip is likely to be very large approaching 0.5 in some cases; this probably induces a large thrust deduction and reduces the hull efficiency below unity.

It is suggested that Taylor's method be used with a real slip of 20 per cent when running free and 50 per cent when towing, making the width of a blade 0.3 the diameter, or even more, and using 2.5 for the value of a. The results can then be compared by calculations for area and thrust as in the preceding paragraph, and by direct comparisons with good examples by methods given on page 491.

Electric Launches.—To allow for light high-speed electric motors, the propellers for launches driven by storage-batteries are run at a very high number of revolutions (300 to 500 per minute). The screws are sometimes two-bladed, and the blades are always narrow and thin.

Taylor's method appears to be well adapted for this purpose; the width of the blade may be made 0.1 of the diameter or less, and the thrust coefficient may be taken at least as large as given by the equations on page 462.

To use Barnaby's method, we may first multiply the electric horse-power by a factor less than unity (0.9 to 0.95) to allow for the greater mechanical efficiency of the motor as compared with a steam-engine, and then multiply by the ratio of the width of the blade to the diameter and divide by 0.2; finally, of course, we allow for using two or three blades instead of four.

Canal-boats.—When canal-boats are driven by a steam-engine or other motor, it is customary to use a wheel with wide blades that may be only partially immersed. Such a wheel when run slowly does not beat the water into foam, as happens when a ship is running light with the propeller partly immersed, though of course the water is somewhat broken. It is suggested that Taylor's method be tried, taking account of only the number of blades immersed. The blades, though wide, need not be thick, as the total power applied is small, for the speed of the boat is limited.

Cavitation.—A screw propeller produces a thrust by imparting velocity to a stream of water, as of necessity all forms of propellers must do. Part of the velocity is attributed to an increase of pressure behind the screw, and part to a diminution of pressure in front of the screw. A certain theoretical investigation by Mr. Thornycroft has led him to think that the ratio of the acceleration of water flowing toward the screw to the total acceleration produced depends on the pitch ratio, and that for large pitch ratios the acceleration toward the screw is larger than half the total acceleration. The whole subject of the accelerations due to the action of a screw propeller is so uncertain that it may be best to admit only that there is an acceleration of water toward the screw and a consequent reduction of pressure, as has already appeared in our discussion of thrust deduction.

When there is a reduction of pressure at a certain place below the surface of the water, there is a flow of water toward that place due to the head of water and to the pressure of the atmosphere; since the latter, for moderate depths, is much the greater (14.7 pounds pressure per square inch), it alone may be taken into account, leaving the head of water for secondary consideration. If we were dealing with water free from air at but little above freezing-point, then it would appear that the limiting reduction of pressure (causing flow toward a place) would be 14.7 pounds per square inch. If the acceleration by a screw propeller is roughly divided half in front of and half behind the propeller, then the limiting thrust that the propeller could pro- duce would appear to be nearly 30 pounds per square inch of projected area. This is, however, far in excess of the real maximum thrust that a propeller can produce, for the water contains air in solution, which is liberated at reduced pressures and breaks the continuity of the stream of water flowing toward the propeller; since sea-water is usually less than 80° F., the effect of vapor formation on account of reduction of pressure is of secondary importance.

From certain tests made on the torpedo-boat-destroyer *Daring*, Mr. Barnaby * assigns 11.25 pounds per square inch as the maximum

* Marine Propellers, 1891.

average thrust that a propeller can produce without danger of breaking continuity of the stream flowing towards the propeller. Some experiments by Charles Parsons in connection with the propulsion of the *Turbinia* confirm this conclusion. Mr. Parsons obtained discontinuity at moderate reduction of pressure by heating the water used for experiments on a small scale, so that vapor of water interrupted the continuity.

This phenomenon is called cavitation, as cavities are formed in front of the propeller; i.e., at the back of the blade.

To calculate the thrust per square inch for the present purpose, divide the total thrust of the screw by the total projected area of the blades of the screw. The total thrust may be obtained from the total resistance (see pages 400 and 406) by dividing it by the thrust factor

$$1-t,$$

which is commonly equal to 0.9.

Or the thrust may be deduced from the horse-power as follows:

First, multiply the I.H.P. by the coefficient of propulsion (0.5 to 0.55) to find the E.H.P. The effective horse-power is then to be divided by the thrust reduction to find the thrust horse-power. Then

$$\text{T.H.P.} = \text{thrust} \times V \times 101.3 \div 33000. \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \text{thrust} = \frac{33000 \times \text{T.H.P.}}{101.3V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the propulsive coefficient be taken as 0.5 and the thrust factor as 0.9, equation (2) gives

$$\text{Thrust} = \frac{33000 \times 0.5\ \text{I.H.P.}}{101.3 \times 0.9V}.$$

$$\text{Thrust in pounds} = 181\frac{\text{I.H.P.}}{V}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If A_p is the projected area in square feet of blades which are just sufficient to avoid cavitation, then

$$\text{Thrust in pounds} = 11.25 \times 144 \times A_p.$$

$$\therefore 11.25 \times 144A_p = 181\frac{\text{I.H.P.}}{V};$$

$$\therefore A_p = 0.112\frac{\text{I.H.P.}}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The relation of the projected area to the developed area depends on the form of the blade and pitch ratio. For the Admiralty blade Barnaby gives the following equation:

$$\text{Projected area} = \frac{\text{developed area}}{\sqrt{1+0.425\ (\text{pitch ratio})^2}}. \quad . \quad . \quad . \quad (5)$$

The Admiralty blade has a width equal to 0.2 of the diameter of the wheel, and the ellipse, if complete, would have an area equal to 0.1 of the area of the disk, as shown on page 439; allowing for the hub, the area of the blade is about 0.09 of the area of the disk.

Now the ratio of projected to developed area of the screw, for which a computation is given on page 477 with a pitch ratio of unity, will be

$$\frac{1}{\sqrt{1+0.425}} = 0.84 \text{ nearly.}$$

The thrust in pounds is $181 \times \frac{3000}{16} = 34000$.

The disk-area $= 162.8$ feet $= 23400$ sq. in.

The projected area of four blades will be

$$4 \times 0.84 \times 0.09 \times 23400 = 7080 \text{ sq. inches.}$$

The thrust per square inch is

$$34000 \div 7080 = 5 - \text{ pounds.}$$

Barnaby's method gives, of course, substantially the same result, for it gives nearly the same diameter of screw. On page 483 the area of the disk was found to be 157.8 square feet instead of that given above, and the thrust per square inch, using this area, appears to be proportionately larger; the difference is insignificant. Barnaby's method is more convenient in other discussions of this subject.

Suppose that we wish to find the smallest screws that can be used to drive the ship at 16 knots with 3000 horse-power without danger of cavitation.

By equation (4) the projected blade area is

$$A_p = 0.112 \frac{\text{I.H.P.}}{V} = 0.112 \frac{3000}{16} = 21 \text{ sq. ft.}$$

If the pitch ratio is unity, then as before the projected area is 0.84 of the developed area, and the latter is consequently

$$21 \div 0.84 = 25 \text{ sq. ft.}$$

The area of one blade is 0.9 of the disk area, so that with four blades the disk area will be

$$25 \div 0.36 = 69.4 \text{ sq. ft.}$$

By Barnaby's equation,

$$\text{Disk area} = 69.4 = C_A \frac{3000}{16^3},$$

$$C_A = \frac{69.4 \times 16^3}{3000} = 94.8.$$

If the pitch ratio is unity as before, this value of C_A will come between columns 11 and 12, and will correspond to nearly 133 for C_R. The diameter of a circle having an area of 69.4 sq. ft. is 9.4 feet; consequently the screw must make

$$133 \times \frac{16}{9.4} = 226$$

revolutions per minute.

The combination of equation (4) with Barnaby's equation for disk area allows us to make a direct calculation for the speed of the ship at which cavitation becomes probable. First let the ratio of the projected area of all the blades to the disk area be represented by r, so that

$$A_p = rA, \quad \text{or} \quad A = \frac{1}{r} A_p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then, from the two equations just mentioned,

$$\frac{1}{r} A_p = A = 0.112 \frac{\text{I.H.P.}}{rV} = C_A \frac{\text{I.H.P.}}{V^3}. \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \; V = \sqrt{C_A \frac{r}{0.112}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This equation has been obtained with the assumption of the following factors:

$1 - w = 0.9, \quad 1 - t = 0.9, \quad$ engine efficiency $= 0.5$,

and takes C_A from the table which applies directly to propellers with four blades. If the factors mentioned differ from the assumed values, then the constant 0.112 will of course vary; but, as already said, refinement appears out of place in this matter.

In the use of Barnaby's tables for designing two- and three-bladed screws we first find a fictitious horse-power by dividing the real horse-power by a factor (0.65 for two blades and 0.865 for three blades) which is equivalent to dividing the function C_A by the proper factor. In order to find the speed to avoid cavitation we may divide C_A by 0.865 for three-bladed screws, and use the result together with the proper value of r in equation (8); as r is proportional to the number of blades (0.75), the speed is less than that possible with four blades; and the speed with two blades is again less than with three.

In the discussion of the use of Barnaby's tables for wide-bladed screws it was suggested that the thrust be assumed to increase with the width, and that we assume a fictitious horse-power which should be less than the real horse-power in the same proportion. Thus if the blade is $0.3D$ instead of $0.2D$, the fictitious horse-power is taken to be two-thirds the real power. This is, of course, equivalent to decreasing the function in like proportion. The effect of this action on the calculation for the speed which avoids cavitation by equation (8) is, however, affected by a proportional increase in the factor r. Consequently this discussion makes it appear indifferent how wide (or narrow) the blade may be. But since the thrust coefficient a in the general theory and in Taylor's method is found to fall off for wide blades, it will be found that when allowance is made for that effect there will be less likelihood of cavitation with wide blades.

As an example find the speed which avoids cavitation with a pitch ratio of unity, using column 5 of Barnaby's table, where

$$C_A = 251.$$

First, as on page 498,

$$r = 4 \times 0.84 \times 0.09 = 0.302.$$

$$\therefore \; V = \sqrt{251 \times \frac{0.302}{0.112}} = 26 \text{ knots.}$$

for a four-bladed screw. For a three-bladed screw

$$r = 3 \times 0.84 \times 0.09 = 0.227.$$

$$\therefore\ V = \sqrt{\frac{251}{0.865} \times \frac{0.227}{0.112}} = 24 \text{ knots.}$$

In looking at Barnaby's tables it appears that the value of C_A increases with the pitch ratio and is greater in the left-hand columns; that is, large slow screws are less likely to be affected by cavitation than small screws with a high speed of rotation. The right-hand columns are likely to lead to the design of screws that give cavitation at even moderate speeds, and in all cases must be carefully guarded from this effect.

While it has been found that increasing the width of the blades of a screw, and decreasing the diameter proportionately, does not affect calculations for cavitation, we may on the other hand avoid cavitation by increasing the width without changing the diameter.

Mr. Barnaby gives 11 inches for the immersion of the upper tips of the blades of the screws of the *Daring*, which screws were guarded against racing by drawing the stern out over them. He says that for each additional foot of depth for the upper tips of a screw we may add three-eighths of a pound allowable thrust per square inch to the standard 11.25. Conversely, if a screw is placed in a tunnel so that its tips are above the level of the water when at rest, a corresponding reduction of thrust should be made. Thus, if the tips are a foot above the normal level, the reduction will be

$$2 \times \tfrac{3}{8} = \tfrac{6}{8} \text{ of a pound.}$$

Twisted Blades.—Separate blades bolted on to a hub have, in addition to other advantages, the further advantage that the pitch may be altered slightly by twisting them. To do this it is necessary to elongate the holes for the bolts which fasten them to the hub; the space left must, of course, be filled with a properly shaped piece of metal so that the blades may be securely held after they have been twisted. It is important to know in advance the angle through which the blade should be twisted to produce a given change of pitch.

If one turn of a helix be developed, it will form the hypothenuse of a triangle having the base πD and the altitude P; the angle at the base may be determined by the equation

$$\tan \alpha = P \div \pi D = p \div \pi. \quad \ldots \ldots \quad (1)$$

If the angle α receives an increment a, the equation (1) becomes

$$\tan (\alpha + a) = p' \div \pi, \quad \ldots \ldots \quad (2)$$

and consequently

$$p' = p \frac{\tan (\alpha + a)}{\tan \alpha}. \quad \ldots \ldots \quad (3)$$

The following table gives the factors by which the pitch-ratio is to be multiplied to find the effect of twisting the blade through small angles:

TABLE FOR TWISTED BLADES.

Angle the Blade is Twisted.		1°.		2°.		3°.		4°.		5°.		6°.	
Pitch Ratio.	Diameter Ratio.	Increase.	Decrease.	Increase.	Decrease.	Increase.	Decrease.	Increase.	Decrease.	Increase.	Decrease.	Increase.	Decrease.
0.8	1.25	1.07	0.93	1.15	0.85	1.22	0.78	1.29	0.71	1.37	0.64	1.45	0.57
0.9	1.11	1.07	0.94	1.14	0.87	1.20	0.81	1.27	0.75	1.34	0.68	1.41	0.61
1.0	1.00	1.06	0.94	1.12	0.88	1.19	0.82	1.25	0.76	1.31	0.70	1.38	0.65
1.1	0.90	1.05	0.94	1.11	0.89	1.17	0.84	1.23	0.78	1.29	0.73	1.35	0.68
1.2	0.83	1.05	0.94	1.10	0.89	1.16	0.84	1.21	0.79	1.27	0.74	1.32	0.70
1.3	0.77	1.05	0.95	1.10	0.90	1.15	0.86	1.21	0.81	1.26	0.76	1.31	0.72
1.4	0.71	1.05	0.95	1.09	0.91	1.14	0.86	1.20	0.82	1.24	0.77	1.30	0.73
1.5	0.66	1.05	0.96	1.09	0.91	1.14	0.87	1.19	0.83	1.24	0.78	1.29	0.74
1.6	0.62	1.04	0.96	1.09	0.91	1.13	0.87	1.18	0.83	1.22	0.79	1.28	0.75
1.7	0.59	1.04	0.96	1.08	0.92	1.13	0.88	1.17	0.84	1.22	0.80	1.27	0.76
1.8	0.55	1.04	0.96	1.08	0.92	1.13	0.88	1.17	0.84	1.21	0.81	1.26	0.77
1.9	0.52	1.04	0.96	1.08	0.92	1.12	0.88	1.16	0.85	1.20	0.81	1.25	0.78
2.0	0.50	1.04	0.96	1.08	0.92	1.12	0.88	1.16	0.85	1.20	0.81	1.25	0.78
2.1	0.47	1.04	0.96	1.08	0.92	1.12	0.88	1.16	0.85	1.20	0.82	1.24	0.79
2.2	0.45	1.04	0.96	1.07	0.93	1.11	0.89	1.15	0.85	1.20	0.82	1.24	0.79
2.3	0.43	1.03	0.96	1.07	0.93	1.11	0.89	1.15	0.86	1.19	0.83	1.24	0.79
2.4	0.42	1.03	0.96	1.07	0.93	1.11	0.89	1.15	0.86	1.19	0.83	1.24	0.80
2.5	0.40	1.03	0.96	1.07	0.93	1.11	0.90	1.15	0.86	1.19	0.83	1.23	0.80

When a propeller-blade has a variable pitch it is customary to take the arithmetical mean of the value of the pitch measured at regular intervals for the pitch of the screw. If the pitch increases

uniformly from the axis, then the pitch at the middle of the length of the blade is taken as the mean pitch. It is evident that twisting a blade affects the pitch much more near the tip than near the hub, and that when the angle is increased the blade has a radially increasing pitch.

Strength of Propeller-blades.—The method developed by Naval Constructor Taylor has been applied by him to the production of a logical means of calculating the stress in a propeller-blade; none of the other methods given in this book is applicable for this purpose.

Returning to equations (5) and (6), page 454,

$$\Delta T=\Delta N \cos\theta-\Delta F \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Delta U=\Delta N \sin\theta+\Delta F \cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting for ΔN, ΔF, and the trigonometric functions their values as in the theory of propellers,

$$\Delta T=\frac{1}{60^2}P^2R^2\Delta A\left\{\frac{as\pi^2d_a^2\sqrt{\pi^2d_a^2+(1-s)^2}}{\pi^2d_a^2+1}-f\sqrt{1+\pi^2d_a^2}\right\}, \quad . \quad . \quad (3)$$

$$\Delta U=\frac{1}{60^2}P^2R^2\Delta A\left\{\frac{as\pi d_a\sqrt{\pi^2d_a^2+(1-s)^2}}{\pi^2d_a^2+1}+f\pi d_a\sqrt{\pi^2d_a^2+1}\right\}. \quad . \quad . \quad (4)$$

Or, introducing the functions α, β, and γ,

$$\Delta T=\frac{1}{60^2}P^2R^2(as\alpha-f\beta)\Delta A, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\Delta U=\frac{1}{60^2}P^2R^2\left(\frac{as\alpha}{\pi d_a}+\frac{f\gamma}{\pi d_a}\right)\Delta A. \quad . \quad . \quad . \quad . \quad (6)$$

Let moments be taken of these forces about axes through the shaft; the turning force will be at a distance r from the axis of the wheel and will have the moment $r\Delta U$ about that axis; the thrust will have a moment $r\Delta T$, but this moment will be about an axis at right angles to the axis of the wheel. The radial distance may be taken to be

$$r=\tfrac{1}{2}D_a=\tfrac{1}{2}Pd_a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where d_a is the diameter ratio.

The moment of the thrust is consequently

$$r\Delta T=\frac{1}{2\times 60^2}P^2R^2(as\alpha d_a-f\beta d_a)\Delta A, \quad . \quad . \quad . \quad . \quad (8)$$

and the moment of the turning force is

$$r\Delta U=\frac{1}{2\times 60^2\pi}P^2R^2(as\alpha+f\gamma)\Delta A. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In order to integrate for a propeller-blade make

$$\Delta A=l\Delta r=D^2\left(\frac{l}{D}\right)\Delta\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which differs from the convention on page 459 in that it allows us to deal with one blade at a time. Putting this blade in equations (5) and (6), (8) and (9), and integrating:

Thrust, $T=\frac{1}{60^2}P^2R^2D^2\left\{as\int\alpha\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)-f\int\beta\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)\right\}$. (11)

Transverse force,

$$U=\frac{1}{60^2\pi}P^2R^2D^2\left\{as\int\alpha\frac{1}{d^a}\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)+f\int\gamma\frac{1}{d_a}\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)\right\}. \quad (12)$$

Moment of thrust,

$$M_t=\frac{1}{2\times 60^2}P^3R^2D^3\left\{as\int\alpha d_a\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)-f\int\beta d_a\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)\right\}. \quad (13)$$

Moment of transverse force,

$$M_u=\frac{1}{2\pi 60^2}\left\{as\int\alpha\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)+f\int\gamma\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right)\right\}. \quad . \quad (14)$$

The integral expressions may be represented by single letters as follows:

$$A=\int\alpha\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$B=\int\beta\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$C=\int r\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$E=\int \alpha\frac{1}{d_a}\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$F=\int r\frac{1}{d_a}\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$G=\int \alpha d_a\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$H=\int \beta d_a\left(\frac{l}{D}\right)d\left(\frac{r}{D}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The equations for forces and moments are consequently:

Thrust, $$T=\frac{1}{60^2}P^2R^2D^2(asA-fB). \qquad (22)$$

Transverse force, $$U=\frac{1}{60^2\pi}P^2R^2D^2(asE+fF). \qquad (23)$$

Moment of thrust, $$M_t=\frac{1}{2\times 60^2}P^3R^2D^2(asG-fH). \qquad (24)$$

Moment of transverse force, $$M_v=\frac{1}{2\pi 60^2}P^3R^2D^2(asA+fC). \qquad (25)$$

The centre of application of the thrust may be located at the distance k_1D from the axis of the wheel, obtained by dividing the moment (24) by the thrust (22):

$$k_1D=\tfrac{1}{2}P\frac{asG-fH}{asA-fB}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\therefore\ k_1=\frac{1}{2d}\,\frac{asG-fH}{asA-fB}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where d is the diameter ratio of the wheel. And in like manner the centre of application of the transverse force can be located at a distance k_2D from the axis of the wheel, obtained by dividing the transverse moment by the transverse force:

$$k_2D=\tfrac{1}{2}P\frac{asA+fC}{asE+fF}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\therefore\ k_2=\frac{1}{2d}\,\frac{asA+fC}{asE+fE}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The functions A, B, and C have already been discussed and tables are given on pages 464 and following, by aid of which they can be readily obtained. The functions E, F, G, and H can be determined graphically in much the same way. The following tables give values for all the functions for blades that have a width of $0.2\ D$:

FUNCTIONS FOR ADMIRALTY BLADE.

Diameter Ratio.	A	B	C	E	F	G	H
0.4	0.026	0.083	0.053	0.101	0.198	0.0072	0.021
0.5	0.037	0.091	0.091	0.119	0.274	0.013	0.026
0.6	0.048	0.099	0.146	0.132	0.365	0.021	0.033
0.7	0.060	0.108	0.216	0.143	0.469	0.031	0.040
0.8	0.073	0.118	0.312	0.151	0.585	0.042	0.050
0.9	0.086	0.128	0.434	0.158	0.715	0.054	0.062
1.0	0.099	0.139	0.585	0.164	0.864	0.069	0.079
1.1	0.112	0.148	0.771	0.170	1.025	0.084	0.103
1.2	0.125	0.159	0.997	0.174	1.199	0.100	0.128

FUNCTIONS FOR TAYLOR'S BLADE.

Diameter Ratio.	A	B	C	E	F	G	H
0.4	0.031	0.095	0.087	0.115	0.222	0.009	0.023
0.5	0.043	0.100	0.117	0.128	0.312	0.015	0.032
0.6	0.056	0.108	0.169	0.143	0.415	0.023	0.042
0.7	0.069	0.118	0.251	0.156	0.525	0.033	0.052
0.8	0.083	0.128	0.367	0.161	0.647	0.046	0.062
0.9	0.092	0.140	0.516	0.173	0.797	0.061	0.081
1.0	0.111	0.152	0.707	0.179	0.976	0.077	0.099
1.1	0.127	0.164	0.962	0.186	1.185	0.092	0.124
1.2	0.141	0.176	1.246	0.194	1.426	0.108	0.149

FUNCTIONS FOR STRAIGHT-EDGED BLADE.

Diameter Ratio.	A	B	C	E	F	G	H
0.4	0.029	0.082	0.063	0.103	0.210	0.0115	0.0221
0.5	0.042	0.091	0.110	0.118	0.355	0.0153	0.0313
0.6	0.055	0.101	0.177	0.130	0.424	0.0219	0.0417
0.7	0.068	0.111	0.266	0.140	0.504	0.0304	0.0532
0.8	0.082	0.121	0.380	0.147	0.605	0.0418	0.0656
0.9	0.095	0.133	0.530	0.154	0.782	0.0561	0.0800
1.0	0.109	0.144	0.720	0.160	0.909	0.0731	0.0965
1.1	0.123	0.156	0.955	0.164	1.127	0.0926	0.1161
1.2	0.137	0.168	1.245	0.168	1.324	0.1155	0.1370

The factors k_1 and k_2 for determining the points of application of the thrust and of the transverse force can be computed for any form of blade by equations (27) and (29) after the proper functions A, B, C, D, E, F, G, and H are determined. These factors k_1 and k_2 are independent of the width of the blade, consequently the values determined for the blade having the standard width may be used for all widths of blade. The following tables give values of k_1 and k_2 for the Admiralty blade, for Taylor's blade, and for the straight-edged blade:

FACTORS FOR LOCATING CENTRES OF THRUST AND TRANSVERSE FORCE; 0.2 WAKE.

Diameter Ratio.	Admiralty Blade.		Taylor's Blade.		Straight-edged Blade.	
	Thrust Factor, k_1.	Transverse Force Factor, k_2.	Thrust Factor, k_1.	Transverse Force Factor, k_2.	Thrust Factor, k_1.	Transverse Force Factor, k_2.
0.4	0.35	0.32	0.35	0.32	0.51	0.35
0.5	0.36	0.31	0.36	0.33	0.37	0.35
0.6	0.37	0.31	0.35	0.32	0.33	0.35
0.7	0.37	0.31	0.34	0.31	0.32	0.35
0.8	0.36	0.31	0.34	0.31	0.32	0.34
0.9	0.35	0.31	0.35	0.30	0.33	0.35
1.0	0.35	0.31	0.35	0.30	0.34	0.35
1.1	0.34	0.31	0.32	0.32	0.34	0.35
1.2	0.33	0.31	0.32	0.32	0.35	0.35

To find the distance of the centre of application of the thrust on a given blade from the axis of the shaft, multiply the diameter of the screw in feet by the proper factor k_1; the centre of application of the transverse force may be located in a similar way by aid of the factor k_2. The values of these factors change but slowly, so that a sufficient location can be made by aid of the brief tables given. Barnaby assumes that both thrust and transverse force are applied at $\frac{1}{3}D$ from the axis, which may be a sufficient approximation, considering the difficulties of determining the real stress on a propeller-blade. There are three ways of determining the total thrust and total transverse force acting on a propeller-blade:

1. The thrust and the transverse force may be determined by the equations (22) and (23).

2. Determine the thrust by equation (22), but to find the transverse force compute the moment of that force by equation (25),

which contains only the functions A and C, and divide that moment by $k_2 D$.

3. Derive both the thrust and the transverse force from the indicated horse-power. Let the mechanical efficiency of the engine be represented by e_m; this quantity varies from 0.8 to 0.9. Let the efficiency of the propeller be represented by e; this quantity varies from 0.6 to 0.7. Then the gross horse-power and the effective horse-power may be computed from the indicated horse-power by the following equations:

$$\text{G.H.P.} = e_m \text{ I.H.P.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\text{E.H.P.} = e \text{ G.H.P.} = ee_m \text{I.H.P.} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The distance through which the thrust of the propeller acts in one second is

$$\frac{1}{60} PR(1-s),$$

where s is the real slip allowing for the wake. If there are N blades, the effective horse-power will be

$$\text{E.H.P.} = \frac{1}{60 \times 550} PR(1-s)NT.$$

$$\therefore T = 33000 \frac{ee_m \text{ I.H.P.}}{PR(1-s)N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The work of a turning moment in one second is equal to that moment multiplied by its angular velocity,

$$\frac{2\pi R}{60};$$

conversely, the turning moment can be calculated from the gross horse-power applied to the propeller-shaft by dividing it by

$$\frac{2\pi R}{60 \times 550};$$

consequently the transverse moment is

$$M_u = \frac{33000 \text{ G.H.P.}}{2\pi RN}, \quad \ldots \ldots \ldots \quad (33)$$

remembering that we are now considering only one of the N blades. The transverse force is consequently equal to

$$U = M_u \div k_2 D = \frac{33000\, e_m \text{ I.H.P.}}{2\pi k_2 DRN}. \quad \ldots \ldots \quad (34)$$

Having the thrust T and the transverse force U in pounds, the stress which they produce in the blade may be computed by an application of the ordinary theory of beams, in which it is convenient to take dimensions in inches. The bending moment of the thrust at the root of the blade when the ***radius*** of the hub is $\frac{1}{n}$ part of the diameter of the wheel will be

$$m_t = T\left(k_1 - \frac{1}{n}\right)D \text{ (in inches)}, \quad \ldots \ldots \quad (35)$$

and the bending moment of the transverse force will be

$$m_u = U\left(k_2 - \frac{1}{n}\right)D \text{ (in inches)}. \quad \ldots \ldots \quad (36)$$

In Fig. 202 let ab represent a section of a propeller-blade near the hub by a plane parallel to the shaft, and let ST be the projection on the same plane of the axis of the shaft. The direction of motion of the ship is from T to S, that is, to the right; the rotation of the propeller is right-handed, or from A toward U. Let T be the thrust and U the transverse force, calculated by equations (32) and (34). These forces will be applied at the centre of thrust and the centre of transverse force at about two-thirds of the length of the blade, and will produce bending moments at the section near the hub that can be determined by equations (35) and (36); let

these moments be represented by m_t and m_u, drawn as customary so as to represent a right-handed effort when viewed from the end toward A. Here the forces T and U are the effects of the resistance or pressure of the water on the blade, and are properly indicated as

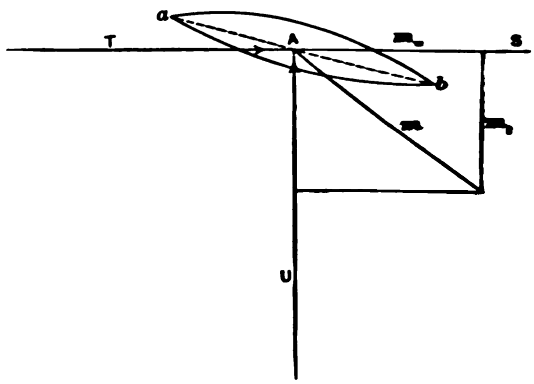

FIG. 202.

being directed toward the blade. The resultant moment m may be calculated by the equation

$$m=\sqrt{m_t^2+m_u^2}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

The stress, which will be a tension at a and a compression at b, is to be found in the usual way by the equation

$$\sigma=\frac{my}{I}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

where σ is the stress per square inch, I is the moment of inertia of the section about an axis parallel to the line Am through the centre of gravity of the section, and y is the distance of a (or of b) from that axis at which the tension (or compression) is the greatest. The bending moment m is in inch-pounds, obtained by taking D in inches as indicated for equations (35) and (36).

In practice it will be convenient to draw the section of the blade on a large scale and to lay off the moments m_t and m_u and to construct the line Am, and draw the axis parallel to this line through

the centre of gravity of the section. The moment of inertia of the section can then be obtained by aid of an integrator; or it may be calculated. If the latter method is preferred it will be found convenient to calculate the moments of inertia about two axes, one parallel to the face of the blade and one at right angles, and to deduce from these the moment of iner ia about the required axis in the usual way. The computation for circular segments is somewhat troublesome, and it will be found easier to substitute for them segments of parabolas like Fig. 203.

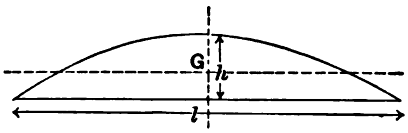

FIG. 203.

If the breadth is l and the height h, then the area will be

$$A = \tfrac{2}{3}lh. \quad \ldots \ldots \ldots \quad (39)$$

The distance of the centre of gravity G from the vertex will be

$$\tfrac{2}{5}h. \quad \ldots \ldots \ldots \quad (40)$$

The major moment of inertia about an axis through G, the centre of gravity, and perpendicular to the face will be

$$I_0 = \tfrac{1}{30}l^3h. \quad \ldots \ldots \ldots \quad (41)$$

The minor moment of inertia about an axis through G parallel to the face will be

$$i_0 = \tfrac{8}{175}lh^3. \quad \ldots \ldots \ldots \quad (42)$$

A section like Fig. 202, bounded by two arcs, may have its moments of inertia computed by aid of these equations as follows: The major moments of inertia for the two parabolic segments may be computed separately by equation (41), using for h in each case the height of the arc from its chord; and the moments of inertia thus computed may be added to get the major moment of inertia of the

section. The minor moments of inertia for each section about an axis through its centre of gravity may be computed separately by equation (42), and then each moment of inertia may be transferred to an axis through the centre of gravity of the section, after which the transferred moments of inertia may be added to get the minor moment of inertia of the section.

If the angle bAm in Fig. 202 is represented by α, then the moment of inertia of the section about an axis through its centre of gravity and parallel to Am will be

$$i = I_0 \sin^2 \alpha + i_0 \cos^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

The thickness of the blade near the tip may be made $\frac{3}{64}$ of an inch per foot of diameter of the wheel, and the blade may be given a straight taper as in Fig. 191, page 446. It is considered that this construction will give sufficient strength at all sections.

The centrifugal tension on a blade of a propeller which has a high speed of rotation is likely to be important, and should always be computed. In the first place the volume, the weight, and the location of the centre of gravity of the blade may be computed by aid of the sections like those of Fig. 192, page 448, when the weight per cubic foot of the material is known. The perpendicular distance c of the centre of gravity of the blade from the axis of the shaft may then be determined, and the centrifugal force may be calculated by the equation

$$F_2 = \frac{w}{g} \frac{4\pi^2 R^2 c}{60^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

in which w is the weight of the material of the blade in pounds per cubic foot, g is the acceleration of gravity (33.2), and R is the number of revolutions per minute.

In addition to the stresses that can be computed, propeller-blades are subject to large and irregular stresses at sea, especially when the pitching of the ship produces racing of the engine, so that the stresses as computed should be kept low, and finally comparison should be made with good practice.

Naval Constructor Taylor recommends the following stresses, not taking account of centrifugal tension:

Material.	Tension.	Compression.
Cast iron	2000	6000
Cast steel	5000	10000
Composition	3000	4000
Manganese or phosphor-bronze	5000	6000

Mr. Barnaby, by following much the same method as that derived from Taylor's method, arrives at the following semi-empirical equation for the thickness of the blade at the hub:

$$h^2 = c\frac{\text{I.H.P.}}{\text{N}}\,\frac{D-d}{b}\left(\frac{d}{PV}+\frac{20}{RD}\right),$$

where D = diameter of the propeller in feet;
d = diameter of the hub in feet;
P = pitch in feet;
R = revolutions per minute;
N = number of blades;
V = speed in knots;
b = breadth of section in inches;
h = height of section in inches;
$c=2$ for steel or manganese bronze;
$c=5$ for gun-metal;
$c=6$ for cast iron.

The bolts which fasten separate blades to the hub should be calculated to resist the moment tending to tear the disk at the root of the blade from its socket in the hub. There are commonly four bolts on one side and three on the other side, as space may allow. The tendency of the bending moment about the axis mm' (see Fig. 204) is to rotate the disk about its edge nn' parallel to mm', which will produce tension on all of the bolts. The distribution of tension among the several bolts is indeterminable, depending largely on the screwing up when they are forced home. If the initial tension due to screwing up the bolts produces a moment about nn' which is greater than the bending moment about

the axis *mm'*, then the initial tension will not be increased by the bending moment. As the bolts are never very large and are well set up, it will probably be best to assume a fair stress per square inch due to screwing up (5000 or 6000 pounds), and then find the moment of the tensions on all the bolts about *nn'* by multiplying the tension on each bolt by its distance from *nn'* and summing up. The same

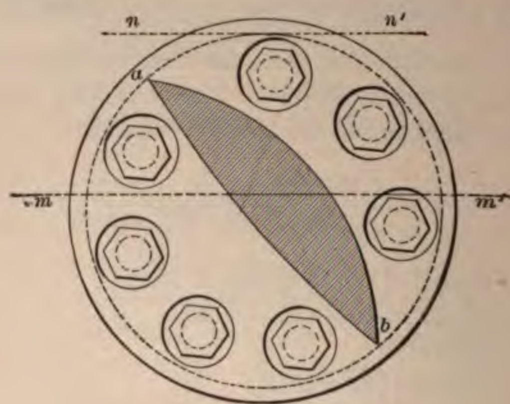

FIG. 204.

result will be obtained by multiplying the tension on one bolt by the sum of the distances of all the bolts.

Screw-turbine Propeller.—A method of propulsion has been invented by Mr. Thornycroft* which appears to be intermediate between screw propulsion and hydraulic propulsion, and which is especially adapted to the propulsion of high-speed shallow-draught boats. The propellers, of which there may be two or three, are arranged much like ordinary screw propellers at the stern of the boat, with horizontal shafts and with the engines toward the middle of the boat. The propellers are of the straight-edged type with three very wide blades which have an axially increasing pitch. The pitch of the leading edge of a blade is designed to correspond with no slip at the normal speed of the boat, and there is a uniformly increasing slip (or acceleration) to the after-edge, and a strong rotation of the race, or the water acted on by the propeller. The screw runs inside

* Marine Propellers. Sidney Barnaby.

a metallic cylinder which confines the race and directs it all into a large number of fixed guide-blades at the after-edge of the cylinder; these guide-blades are set to receive the rotating race without shock, and are curved so as to check its rotation and deliver it in a uniform stream parallel to the shaft. From this action the guide-blades receive a thrust half as great as that exerted by the screw, or one-third the propulsive thrust of the device. The propeller-hub has an increasing diameter from the leading edges of the blades to the after-edges, so that the effective channel for the flow of water is gradually contracted as the axial velocity of the race increases. The fixed hub for the guide-blades is a continuation of the form of the propeller-hub, and abaft the guide-blades there is a tapering tail to prevent eddying. When a boat has a single-screw turbine propeller this tail is combined with the rudder and moves with it. The propellers are placed so that a quarter of the diameter is above the water when the boat is at rest, and the stern is shaped so as to form a tunnel for each propeller which shall lead water up to it and down below the water-line aft. When a propeller is started it expels the air from its tunnel, and then works in unbroken water. Access can be had to the propeller when at rest through a manhole in the top of its tunnel. This arrangement has later been applied to ordinary screw propellers, and the same general idea appears in the broad shallow stern over the screws of a torpedo-boat.

Mr. Barnaby says that the chief advantage of the propeller lies in the fact that its maximum efficiency, which is equal to that of the ordinary screw, is obtained with a very high slip. Thus, while the thrust at maximum efficiency of an ordinary three-bladed screw one foot in diameter is 12 pounds, the thrust of a screw turbine of the same diameter and efficiency is 26½ pounds, and with somewhat less efficiency the thrust may be 32 pounds. Incidentally the speed of rotation for a given power is high, which is favorable for light craft.

The description has taken as its basis a screw propeller, indicating the modifications to change it into a screw-turbine propeller. It is instructive to consider the device as a direct-flow turbine pump applied to hydraulic propulsion. To avoid large passages inside the hull, and also bends and valves, the pump is placed outside the hull

proper and with a partially enclosed tunnel leading the water from the pump; this allows the use of a large pump and better fluid efficiency than can be obtained by other forms of hydraulic propulsion.

A serious defect of this form of propeller is the fact that it has very little effect when reversed to back the boat. To provide for backing, an ordinary four-bladed screw is placed on the shaft forward of the screw turbine. It has the same pitch as the leading edge of the turbine, and is expected when running forward to pass the water without rotating it or producing any thrust; the frictional resistance of this screw is of course a direct loss, but it is not considered to be important. When the engine is reversed this auxiliary screw backs the boat sufficiently for manœuvring.

CHAPTER XII.

POWER FOR SHIPS.

One of the most important problems in the design of a steamship is properly to proportion the power to the speed desired. If, on the one hand, a deficiency in power leads to a disappointment in that the desired speed cannot be attained, on the other hand the provision of power enough to give an unnecessary speed may seriously interfere with the commercial success of the ship, since it must be obtained at the expense of the carrying capacity of the ship; and further, an engine which is worked at much less than its proper power is likely to be very uneconomical.

The determination of the power for a merchant-ship is a comparatively simple matter because the ship is expected to steam always at full power and full speed. The provision of power for a war-ship is much more difficult, not only because war-ships are given relatively high speeds considering their size, length, and form, but because they are expected to cruise at much reduced speeds. Engines for war-ships are commonly smaller and lighter, and run at a higher speed than those for merchant-ships; at full speed the boilers are forced harder and the engine has less expansion, both of which interfere with economy. At a reduced speed the boilers may be worked economically, and at cruising speed the number of boilers in use can be adjusted to the power. Sometimes the economy of the engine can be improved for reduced speed by shortening the cut-off and increasing expansion, but the proportion of cylinders is not well adapted to give the best economy. The engines are always disproportionately large to give economy at cruising speeds, so that the steam and coal consumption at such speeds is comparatively large.

Twin-screw boats sometimes cruise with one engine only in service, but unless the idle screw can be set free to revolve without its engine, the drag of that screw interferes with the contemplated gain. Triple-screw ships may cruise with the middle screw only in service, the two wing screws in that case being released to revolve freely. Some ships, like the U. S. S. *New York*, have been given four engines, two on each shaft; for cruising the after-engines are used, and for full speed the forward engines also are coupled up.

As many as five methods of determining the power required for a ship may be distinguished, but some of the methods are so related that they cannot be considered as entirely independent. They are: (1) independent estimate; (2) model tank method; (3) mechanical theory of similitude; (4) Admiralty coefficient; (5) Kirk's analysis. The last-named method was devised for use with slow freight-ships and has a restricted use. Of the other methods the first has the greatest flexibility for ordinary use, the second requires extensive apparatus, and the third and fourth must be carefully guarded to avoid serious error. Any method may give good results in the hands of a skilful designer who has sufficient information.

Independent Estimate.—To apply this method, the wave-making resistance of the ship, and also the surface or frictional resistance, are computed separately, the efficiency of the propeller, the mechanical efficiency of the engine, and the hull efficiency are estimated, and from these elements the power is determined. The wave-making resistance can be calculated by aid of equation (1) or equation (2), on page 400; the frictional resistance may be calculated by aid of equation (1), page 406. Taking the first form for the wave-making resistance, the total or tow-rope resistance may be written

$$R = fSV^n + b\frac{D^{\frac{2}{3}}}{L}V^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R is the resistance in pounds, S is the wetted surface in square feet, L is the length of the ship in feet, D is the displacement in tons, V is the speed in knots per hour, and f, b, and n are functions for which the values are to be sought in tables as indicated in the discussion of resistance.

One knot per hour is equal to

$$6080 \div 60 = 101.3$$

feet per minute. Consequently the effective horse-power requured to overcome the tow-rope resistance is

$$\text{E.H.P.} = \frac{101.3}{33000} RV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \text{ E.H.P.} = 0.00307\left(fSV^{n+1} + b\frac{D^{\frac{2}{3}}}{L}V^5\right). \quad . \quad . \quad . \quad (3)$$

The propeller efficiency may be determined by Taylor's method using equation (39), page 461, or it may be taken from Barnaby's table, page 484, if the propeller is designed by his method; the efficiency of a well-designed propeller will be between 0.50 and 0.70. The engine efficiency may be estimated between 0.8 and 0.9. The hull efficiency is represented by the expression (18), page 472:

$$\frac{1-t}{1-w},$$

where t is the thrust deduction and w is the wake factor. The hull efficiency can in general be determined only from experiments on models with the propellers in place; it is very nearly unity for large well-formed ships. If the propeller efficiency is represented by e and the mechanical efficiency of the engine is represented by e_m, then the relation of the effective horse-power to the indicated horse-power is given by the equation

$$\text{E.H.P.} = ee_m\frac{1-t}{1-w}\text{I.H.P.}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and conversely,

$$\text{I.H.P.} = \frac{1}{ee_m\frac{1-t}{1-w}}\text{E.H.P.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that

$$\text{I.H.P.} = \frac{0.00307}{ee_m\frac{1-t}{1-w}}\left\{fSV^{n+1} + b\frac{D^{\frac{2}{3}}}{L}V^5\right\} \quad . \quad . \quad . \quad . \quad (6)$$

Model Tank.—The construction of model tanks and the method of using results from experiments on models are given on pages 415–419. It is customary to determine the hull efficiency also in such tanks, so that the effective horse-power and the coefficient of propulsion may both be obtained from the results of experiments.

A model tank is particularly valuable in the investigation of new forms and the solution of new problems. It is, however, absolutely essential that tank experiments should be verified and controlled by tests on full-sized ships. Moreover, tests in a tank are comparable to speed trials in quiet water, and experience has shown that the best form of hull for sustained speed in rough water differs materially from that which gives the highest speed in quiet water. The model tank method can be used only by those few naval architects and constructors who have direct access to results of tank experiments.

Theory of Mechanical Similitude.—This method is often known as Froude's extended law of comparison. In the discussion of the theory of mechanical similitude, it appears that ships which have speeds proportional to the square root of their lengths have their engine powers proportional to the $\frac{7}{6}$ powers of their displacements. Such ships will have the same hull efficiency, and the engine efficiency can safely be assumed to be the same; consequently, if D_1 and D_2 are the displacements,

$$(\text{I.H.P.})_1 : (\text{I.H.P.})_2 :: D_1^{\frac{7}{6}} : D_2^{\frac{7}{6}}; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the lengths have the proportion

$$L_1 : L_2 :: \sqrt[3]{D_1} : \sqrt[3]{D_2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

consequently the speeds, which are as the square roots of the lengths, will have the relation

$$V_1 : V_2 :: D_1^{\frac{1}{6}} : D_2^{\frac{1}{6}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This method may be used directly to determine the power for a new ship which is similar to a known ship and does not differ much in size; but since the coefficient of friction for long ships is less than for short ships, there will be a tendency to overestimate the power of a large ship which is designed from data taken from a small ship, and vice versa.

If the speed of the new ship differs somewhat from the relation given by proportion (9), then the power for the corresponding speed may be found by the proportion, (7) and afterwards the power may be assumed to vary with the cube of the speed for moderate changes of speed.

From the table for friction constants on page 406 it appears that the exponent n of equation (3), page 519, has for its average value 1.83; and consequently the power required to overcome the frictional resistance increases as the 2.83 power of the speed, which is appreciably less than the cube. On the other hand the power required to overcome the wave-making resistance increases as the fifth power of the speed; but that power is usually less than one-third of the total power for the ship. This investigation justifies the above assumption that for moderate changes of speed the power varies as the third power of the speed. The power for torpedo-boats and other vessels that are driven at relatively high speeds is likely to increase more rapidly than the cube of the speed. A successful designer of such vessels must have much experience and very complete results of the trials of boats of the class.

For the application of the theory of mechanical similitude there should be at hand tables of data of successful ships like those on pages 522 and 523. It is convenient to reduce the data to a standard displacement of 10000 tons by the proportions (7), (8), and (9); the standard displacement for torpedo-boats and other small craft may be smaller than that given for large ships if desired.

Admiralty Coefficients.—Taking the proportion (7),

$$(\text{I.H.P.})_1 : (\text{I.H.P.})_2 : : D_1^{\frac{7}{6}} : D_2^{\frac{7}{6}},$$

which expresses the conditions from the theory of mechanical similitude for ships which have speeds proportioned to the square roots

DATA FROM VARIOUS SHIPS.

	Name.	Length between Perp.	Beam.	Trial Draft.	Trial Disp't.	Block Coeff.	Wetted Surface.	Speed.	I.H.P.	I.H.P. per Ton.	Reduced to 10,000 T. Disp.			
					Tons		Sq. Ft.	Knots			Adm. Coeff.	I.H.P.	Speed.	Length.
High-speed passenger.	* Deutschland	660–9	67–0	28–10	23,200	.631	59,200	23.5	35,500	1.53	207.0	13,300	20.42	500.4
	Campania	600–0	65–3	25–0	18,000	.64	49,620	23.18	31,050	1.72	270.0	15,018	21.02	402.0
	Korea	550–0	63–0	27–0	18,400	.69	48,040	20.0	17,000	.974	312.0	8,700	18.07	448.8
	* Kaiser Wilhelm II.	680–0	72–0	29–6	26,500	.644	66,040	23.5	40,000	1.510	288.4	12,810	19.98	400.0
	Savoie	557–8	60–0	25–6	15,400	.634	44,260	21.5	21,700	1.410	283.0	13,110	20.0	483.0
	* Oceanic	685–0	68–5	32–6	28,500	.656	68,830	20.7	27,000	.95	306.8	7,057	17.30	483.1
Intermediate passenger and freight.	Umbria	402–0	46–7	16–4	6,050	.698	30,560	15.7	4,650	.770	270.5	8,380	17.00	474.4
	* Celtic	680–9	75–5	36–6	37,700	.704	78,880	......	14,000	.371	......	2,080	...	417.5
	* Minnesota	608–0	73–0	33–0	33,000	.788	69,600	14.0	10,000	.303	282.4	9,483	11.38	408.0
	Bayern	390–4	45–0	24–10	7,070	.605	26,060	14.0	3,500	.495	287.4	5,257	14.83	411.4
	* Minnetonka	600–0	65–0	33–2	26,530	.719	62,360	16.0	11,000	.410	332.1	3,522	13.60	433.1
Coasters.	J. S. Whitney	272–0	43–0	14–1	2,792	.585	14,027	14.5	2,230	.800	270.8	9,848	17.01	415.6
	* Monroe	344–0	46–0	18–0	5,375	.655	21,090	15.0	4,500	.837	231.5	9,100	16.01	422.5
	Boston	248–0	36–0	13–3	1,746	.515	10,153	14.0	3,091	1.770	131.4	23,800	18.89	443.0
	Gloucester	272–0	42–0	16–4	3,220	.555	14,350	14.5	2,800	.875	236.9	10,500	17.50	307.7
	Howard	272–0	42–0	17–9	3,224	.558	14,684	13.0	1,770	.548	270.1	6,038	15.60	307.7
Freighters and tramps.	* Shawmut	488–0	58–0	27–3	17,200	.78	45,130	12.0	4,000	.233	287.0	9,125	10.96	406.5
	Inchmarlo	360–0	48–0	22–10	8,898	.80	27,890	9.0	1,440	.162	222.7	1,050	9.18	374.0
	Pennsylvania	430–0	50–0	17–4	8,926	.83	30,520	9.89	1,455	.163	286.0	1,001	10.08	416.6
	Delaware	345–0	44–0	24–6	8,200	.77	26,490	12.27	2,680	.327	270.7	3,387	12.60	368.5
	....	370–0	41–0	18–11	4,635	.568	20,390	13.8	2,500	.539	291.6	6,145	15.07	477.4

* [illegible]

Yachts.	Capercailzie.	183–10	24–0	12–2	581	.38	5,100	13.02	600	1.220	249.2	20,430	22.22	473.3
	* Prinzessin Vict. Luise	400–10	47–0	16–6	5,650	.638	23,400	16.0	4,000	.784	258.1	7,788	17.60	484.0
	Wacouta.	208–0	32–0	12–5	1,000	.432	7,073	13.0	1,120	1.120	213.2	16,450	19.08	448.1
	Freelance.	108–0	18–8	7–0	134	.344	2,202	13.0	330	2.46	174.3	50,530	26.68	4[illegible]6.3
	Fauvette.	160–0	22–8	12–11	491	.356	4,590	15.6	1,150	2.34	205.5	38,700	25.78	407.7
Side-wheelers	City of Erie.	314–0	44–0	10–3	2,233	.551	12,776	18.9	6,472	2.90	225.1	37,160	24.28	520.0
	Tashmoo.	300–0	37–6	8–3	1,224	.463	8,976	18.85	3,400	2.79	223.9	39,470	26.70	604.8
	Uncatena.	179–4	31–4	6–3	608	.615	5,535	13.0	982	1.63	160.0	25,760	20.71	455.9
	Gay Head	204–6	34–9	6–9	744	.596	6,534	14.0	1,113	1.50	202.7	23,060	21.60	486.0
Battle-ships.	Kentucky	368–1	72–3	23–6	11,538	.643	33,700	16.9	12,082	1.05	203.3	9,985	16.5	350.8
	Massachusetts	348–0	69–3	24–1	10,265	.619	29,610	16.2	10,127	.987	198.7	9,823	16.14	345.0
	Alabama.	368–0	72–0	24–0	11,734	.644	32,460	17.0	11,200	.958	225.5	9,[illegible]30	16.55	
	Maine.	388–0	72–3	23–8	12,500	.654	34,490	18.0	15,200	1.220	205.8	11,[illegible]0	17.32	360.1
Fast cruisers.	Columbia	411–8	58–2	22–5	7,350	.48	27,990	22.8	17,990	2.45	248.9	25,763	2[illegible].0	456.0
	H.M.S. Leviathan	500–0	71–0	27–0	14,250	.52	41,950	22.4	25,572	1.79	258.7	16,820	21.13	444.3
	H.M.S. King Alfred	500–0	71–0	26–6	14,100	.525	....	23.47	30,950	2.12	233.6	20,740	22.32	446.0
Torpedo-boats.	De Long.	175–0	17–7	5–10	196	.382	....	24.5	2,500	12.75	197.8	508,800	47.3	653.2
	Wilkes	175–0	17–8	4–6	205	.516	....	24.97	3,495	17.05	149.9	329,200	44.4	559.1
	Bagley	157–0	16–0	4–10	168	.485	....	29.15	3,920	23.18	192.4	526,200	58.3	626.5
	Stockton.	175–0	17–8	4–6	196	.441	....	26.03	3,375	17.25	175.7	686,800	50.4	653.2
Torpedo-boat destroyers.	Preble.	244–0	23–6	7–3	474.7	.399	....	27.55	7,110	14.98	179.0	251,000	45.9	698.1
	Truxton.	258–6	22–6	7–11	509.8	.387	....	27.67	7,280	14.30	185.7	236,100	45.6	697.5
	Decatur	250–0	23–8	7–0	450.0	.380	....	28.10	7,300	16.21	178.5	275,600	47.2	706.1
	Hopkins.	244–0	24–4	7–0	452.0	.383	....	27.77	8,866	19.71	142.2	334,700	46.6	689.1

* Designed.

of their lengths, we may first separate the terms displacements and obtain the expression

$$\text{I.H.P.} \propto D^{\frac{2}{3}} \cdot D^{\frac{1}{3}},$$

and then replace $D^{\frac{1}{3}}$ in terms of the speed, obtaini

$$\text{I.H.P.} \propto D^{\frac{2}{3}} V^{3}.$$

Introducing a factor in the denominator, we may w

$$\text{I.H.P.} = \frac{1}{K} D^{\frac{2}{3}} V^{3}. \quad . \quad . \quad .$$

This is a common and convenient way of expressi[illegible] quired for a ship, D being the displacement in tons knots per hour, and K is known as the Admiralty c[illegible] coefficient varies with the form, size, and speed of th[illegible] in all cases be derived from trials of ships similar to [illegible] the method is applied. It is clear that this method theory of mechanical similitude and has the limitatio[illegible] the method. The preceding tables of data of sh[illegible] Admiralty coefficient.

[illegible] Admiralty coefficient method of determining [illegible] and the coefficient is known fo[illegible] [illegible] method, which at first sight seems [illegible] give very good results for ships of us[illegible] [illegible] use an equation which fu[illegible] [illegible] though it appears to differ [illegible] Since the displacement is prop[illegible] [illegible] and the midship section is prop[illegible] [illegible] for similar ships it is clear that [illegible]

$$\text{I.H.P.} = \frac{1}{C} A V^{3}, \quad . \quad . \quad . \quad .$$

[illegible] midship section in square feet, V is the sp[illegible] [illegible] which depends on the size, form, and

ship. Equation (11) seems to make the power required to drive the ship independent of the length and consequently of the dislpacement, but since the equation should be used only for inferring from a known ship the power required for a similar ship, it can be controlled as well as equation (10), which introduces the displacement directly.

Kirk's Method.—Dr. A. C. Kirk proposed a method of determining the power for slow freight-ships which appeared to be so promising that at one time its extension was advocated to all kinds of ships; now this system, known as Kirk's analysis, is little used for ships of any kind.

To use the system it is assumed that the power required for a ship is proportional to the cube of the speed, and that it varies as the wetted surface; this makes this method fundamentally similar to the Admiralty coefficient method, and requires that it shall be controlled with the same care and skill. We may then write

$$\text{I.H.P.} = \frac{kSV^3}{100000}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where S is the wetted surface in square feet and V is the speed in knots, while k is a factor which varies from 4 to 6, according to the size, shape, and speed of the ship. The value 4 is for fine ships with clean bottoms and high efficiency, the value 5 for ships of the ordinary type passenger-steamers, while 6 is for bluff cargo-ships.

Dr. Kirk associated with his method a way of his own for determining approximately the wetted surface of a ship; but his way is less accurate and less convenient than methods given on page 409, which can be properly used with equation (12) if desired.

Example.—To exhibit the use of the several methods let us calculate the power for a cruiser of 13500 tons to make 23 knots, using data derived from the U. S. S. *New York*. The new ship. if similar to the *New York*, will have the length, breadth, and draught derived by the proportion

$$\begin{aligned} (8480)^{\frac{1}{3}} : (13500)^{\frac{1}{3}} &:: 380 \cdot L = 444 \text{ feet,} \\ &:: 64.25 \cdot B = 75 \quad \text{"} \\ &:: 23.9 \quad D = 27.9 \quad \text{"} \end{aligned}$$

and the wetted surface by the proportion

$$(8480)^{\frac{2}{3}} : (13500)^{\frac{2}{3}} :: 28320 : x = 38600 \text{ sq. ft.}$$

1. Assuming an efficiency of 0.65 for the propeller and 0.85 for the engine, and unity for the hull efficiency, and taking 0.4 for the factor for wave-making resistance, and also taking the proper factors for friction from the table on page 406, we have for the indicated horse-power by equation (6) of the method of independent estimate

$$\text{I.H.P.} = \frac{0.00307}{0.65 \times 0.85} \left\{ 0.00926 \times 38600 \times 23^{2.83} + 0.4 \frac{(13500)^{\frac{2}{3}}}{444} 23^{5} \right\} = 32450$$

2. The mechanical theory of similitude gives for the power at the corresponding speed

$$\text{I.H.P.} = 20540 \left(\frac{13500}{10000} \right)^{\frac{7}{6}} = 29140,$$

the corresponding speed being

$$21.59 \left(\frac{13500}{10000} \right)^{\frac{1}{6}} = 22.7 \text{ knots;}$$

consequently the power for 23 knots will be

$$29140 \left(\frac{23}{22.7} \right)^{3} = 30300 \text{ horse-power.}$$

3. The Admiralty coefficient gives

$$\text{I.H.P.} = \frac{1}{227.2} (13500)^{\frac{2}{3}} 23^{3} = 30360,$$

as it should be, since it involves the same principle as the previous method.

In comparing the results by the different methods it is proper to call attention to the fact that 0.4 is probably a large value to give to the factor b in the independent estimate.

Power and Speed Trials.—In order that any of the preceding methods may be used successfully the designer must have sufficient

reliable data from trials of the power and speed of ships of the same class as the proposed ship.

Too much emphasis cannot be given to the necessity for correct and complete power and speed trials of ships. Such trials may be of two kinds: (1) full-power trials lasting several hours, supplemented in some cases by trials at reduced power by the use of natural draught only, by running at lower steam-pressure, or with more expansion, etc.; (2) progressive speed trials over a measured mile.

Full-power trials (and supplementary reduced-power trials) are made at sea, though sometimes they are adjacent to the coast to facilitate the measurement of distance. They may last four or eight hours, or even more, and are intended to show the power of sustaining speed, and are consequently a fair trial of the engines and boilers. Since it is customary to take advantage of good fuel stored in convenient bunkers, and of a numerous and skilled complement of men in both engine- and boiler-rooms, the full-power trials usually show considerably more speed than the ship can maintain for a long period of time. This is especially true of war-ships, but large liners in fair weather will probably make nearly as high speed as can be obtained on any trial.

Full-power speed trials of the U. S. Navy ships are made off the coast on a line marked by buoys set for the purpose, and the tidal connections are obtained from observations made on ships anchored near the buoys.

Sometimes the distance is determined by the number of revolutions of the propeller after progressive speed trials have been made to find the relation between the speed of the ship and the number of revolutions of the screw. This preliminary progressive speed trial is called standardizing the screw.

When a ship is steaming regularly in fair weather at sea the distances run can be determined from the observations of the navigator. It is sometimes proposed to determine speeds by the aid of a patent log; but this method does not command confidence at the present time, though some recent forms of such logs appear to be reliable after they have been rated.

The determination of the speed of the ship does not present serious difficulty to competent observers who have adequate means at com-

mand. The determination of the power by the aid of indicators is much less satisfactory. Under favorable circumstances the unavoidable error of a steam-engine indicator is likely to be from two to five per cent. If the error is to be not more than two per cent., the instrument must be in good condition, must have been properly rated, and must be used with skill; the greatest difficulty comes with fouling of the indicator cylinder, and to avoid such fouling and consequent excessive friction the cylinder must be frequently cleaned. If the run is from four to eight hours long, and if the steam-pressure and other conditions are kept uniform, it will be sufficient to take diagrams at intervals of ten or fifteen minutes. If a voyage or a part of a voyage is taken as the run, then diagrams should be taken often enough to get a proper average of the power; once an hour is recommended. Too often diagrams are taken at infrequent intervals and at times when the conditions are favorable and the power is greater than the average. Thoroughly satisfactory data for a day's run are to be preferred to less reliable data from an entire voyage.

Progressive Speed Trials.—Trials over a measured course at various speeds give the most complete information concerning the power and speed of a ship; but such trials do not test the endurance of the ship's speed, and should be considered as supplementary to full-power trials over long courses.

A measured mile is usually just one nautical mile (6080 feet) long, and is marked by range-poles at the ends; sometimes the statute mile (5280 feet) is chosen instead, or may be marked in addition; any known distance will answer, though less convenient.

If possible, the course should be about a quarter of a mile (500 yards) from the shore on which the poles are set, and the inshore poles should be set about 300 yards back from those near the water. The poles should be painted white or otherwise marked, so that they can be readily distinguished. It is desirable that the course shall be determined by poles set for the purpose or by landmarks already existing; if the course is not marked, the course is run by the compass, but that is less satisfactory. If the ship does not run a true course, the real distance will be greater than the measured distance, and there is likely to be a drag at the rudder due to changing course.

It is desirable that the measured mile shall be located where there is little if any current from tides or other action; this is seldom possible on the seacoast. In any case the course should be free from cross-currents; the influence of direct currents can be eliminated by running in both directions, one or more pairs of runs being made at each speed. It is convenient to take advantage of slack water near the change of tide for runs at slow speed, but it is in general advisable to avoid making a run at the time when the tide changes.

The observations to be taken are (1) the time on the course, (2) the number of revolutions, (3) the mean effective pressure.

(1) The time is to be taken with a stop-watch, which is started when the range-poles at the beginning of the course come into line, and stopped when the poles at the end of the course come into line.

(2) The number of revolutions for the run is to be taken from the engine-counter (when there is one), which is read at the beginning and at the end of the course, on a signal from the observer on deck who is taking time. For this purpose an electric bell should be set up near the engine-counter and connected to a push-button or circuit-switch that may be carried by the timer. If the engine has not a counter, and if one cannot be attached, then the revolutions per minute must be counted at the beginning and at the end of the course; if the speed is small, the revolutions may be counted at intermediate times. The revolutions during the trials of a torpedo-boat or other high-speed boat may be conveniently determined by a special form of counter driven by a worm on the engine-shaft; this counter can be made so that the timer can throw it in at the beginning of the run and disengage it at the end, thus eliminating the uncertainty of giving a signal to a separate observer. These boats commonly have two screws which should be driven at the same speed; for this purpose there may be a device based on a train of epicyclic bevel-gears to detect a difference of speeds; one engine may run under constant steam-pressure, and the other may be kept at the same speed by an attendant at its throttle-valve.

(3) Marine engines are usually piped for indicators with a three-way cock; if $\frac{3}{4}$-inch pipe is used, this method is nearly as good as to have separate indicators for each end of the cylinder. At any rate

the indicated horse-power for ships will be relatively correct if all are indicated in the same way. For compound and triple engines the number of indicators required, with a proper number in reserve, will often tax the resources of the party making the trials. Since the time required for the ship to run a mile will be 2 to 12 minutes, it will be necessary to take diagrams expeditiously while the ship is on the course, in order to get a proper number (from 4 to 6). The time required for the turn before making the next run will be 3 to 10 minutes, which can be increased if there is reason. There is time for making slight adjustments of the indicator during a turn, but no time for such work during a run. The time for making a series of trials will be three to six hours, during which the indicators are in nearly continuous service. After an indicator has been in use an hour or so it is likely to become foul and to show excessive friction. The best way is to have a spare indicator ready for each cylinder with the cord adjusted so that it can be instantly put in place when required; if the diagrams for a run show signs of sticking, the reserve indicator should be substituted before the next run. The high-pressure cylinder is most likely to give trouble in this way, and the low-pressure is least likely. With a triple engine it is well to have one man detailed to care for the indicators and immediately put in order each indicator that is removed for any cause; he can also take care of the diagrams and place them in envelopes prepared for the purpose.

Runs should be made at five speeds, one at full speed, one at half speed, and the others spaced at equal intervals as nearly as may be; one or more pairs of runs are made at each speed. The lowest speed and the highest speed should be well determined by a sufficient number of runs, as they are the most important. There is little if any use in indicating the engine at less than half speed, for the power is then only one-eighth of the full power and the mean effective pressures are small and cannot be determined at satisfaction. As a rule fairly stiff springs are preferred for the indicators, as they are less influenced by friction due to fouling; of course the spring must be of the proper scale for the pressure.

Especially for the first trials on a ship when the crew or the observers, or both, must learn their duties, it is well to begin with the

slowest speed. The engineer is to be instructed to set the engine to run the proper number of revolutions as nearly as may be, determining the boiler-pressure and setting the throttle-valve; after the engine has been so set the conditions should be left unchanged till all the runs at that speed are finished, care being taken to keep the boiler-pressure constant. The ship will then run very regularly and satisfactory results are readily obtained. When the course is finished the ship should be swung sharply off for the turn, which should be long enough to give half a mile or a mile on the course before the first range-poles come in line, in order that the ship may come to uniform speed before the time is taken. During the turn the ship will slow down, but will come to the desired speed after it is brought to a straight course again. After a sufficient number of runs have been made at a given speed, the engine may be set in like manner for the next speed. It is likely that when the ship approaches full speed the master of the ship will desire to reduce speed during the turns, and it may be necessary to do so if the trials are run in narrow waters.

In preparation for the full-speed runs it is well to take a recess to allow the fires to be cleaned and to get everything in best condition; meanwhile the ship may be run on or off the course to use up the steam. When ready the full-speed trials will be run with the throttle wide open and with the maximum steam-pressure. No attempt at regular speed should be made except that which comes from the fact that engine and boilers are expected to do their best. It is almost always necessary to slow down somewhat for turns, but it is well to run as fast as may be Though there may be an opinion to the contrary, the best result, even for a mile, is obtained by careful and regular work and not by "jockeying."

If thought desirable, the full-speed trials can be run early in the series, as soon as the fires are in the proper condition, and the other runs may be made afterwards, provided the crew and observers are familiar with their duties.

Analysis of Progressive Speed Trials.—The object of progressive speed trials is to determine the distribution of power and to investigate other conditions of the ship. Such an analysis may proceed as follows, using as illustration the progressive speed trials made on the

U. S. S. *Manning* at Southport, Maine, over a course which ap- to be free from tidal currents, so that tidal corrections, which will discussed later, were not required.

The ship had been docked and cleaned just before the trials; her copper sheathing was in good condition. The measured at Southport is in deep water near shore, and is marked by wh poles against a dark background. The day was quiet and slig overcast. Trials at natural draught only were made.

The principal dimensions * of the *Manning* are as follows:

Length over all	205 feet 6 inches
Length between perpendiculars	188 "
Moulded beam	32 "
Maximum beam	32 " 10 "
Mean draught on trial	12 " 4 "
Displacement on trial	1000.7 tons
Wetted surface	7273 square feet
Diameter high-pressure cylinder	25 inches
" intermediate cylinder	37½ "
" low-pressure cylinder	56¼ "
" piston-rods	5 "
Stroke, all pistons	30 "
Diameter of propeller	11 feet
Diameter of propeller hub	1 foot 10½ inches
Pitch of propeller	12 feet 4 "

The data from the trials are given in the table on page 533.

The first three columns of this table require no explanation; the fourth gives the mean effective pressure of the steam on the pistons of the engine reduced to the low-pressure cylinder. Having the mean effective pressure from the indicator diagrams taken at each end of each cylinder, the reduced mean effective pressure in the table is obtained as follows. The mean effective pressure for the lower end of the high-pressure cylinder is multiplied by the effective area of the lower side of the high-pressure cylinder, allowing for the piston rod, and the product is divided by the mean area of the low-pressure

* Trans. Soc. Nav. Archts. and Marine Engrs., vol. 7.

piston, that is, by the area of one side of that piston minus half the area of the piston-rod; the mean effective pressure for the upper end is treated in the same way except that no allowance is made for the piston-rod; the mean effective pressures, for both ends of the intermediate cylinder and of the low-pressure cylinder are reduced in a similar way; finally, all the reduced pressures are added to get the reduced mean effective pressure for the engine as recorded in the table.

PROGRESSIVE SPEED TRIALS OF THE U. S. S. MANNING.

No. of Run.	Revolutions per Minute.	Speed in Knots per Hour.	Mean Effective Pressure.
1.	2.	3.	4.
1	58.3	6.78	6.05
2	55.5	6.45	5.70
3	61.2	7.07	6.29
4	62.0	8.05	6.38
5	82.7	9.65	10.01
6	78.5	9.08	9.45
7	111.9	12.74	19.30
8	113.7	12.60	19.82
9	115.5	12.88	20.31
10	114.1	12.65	19.84
11	147.9	15.63	35.06
12	142.0	15.00	33.99
13	152.0	16.00	38.26
14	151.6	15.85	37.76

On Fig. 205 two curves are drawn, both having the number of revolutions per minute for abscissæ. One curve, which is known as the speed curve, has the speeds taken from the preceding table as ordinates; the other has for its ordinates the logarithms of the reduced mean effective pressure from the same table. The speed curve passes through the origin because the vessel will remain at rest if the engine is not running. This curve is nearly straight at the lower end, but shows that the speeds do not increase as rapidly as the revolutions for the higher speeds, which indicates that there is a larger apparent slip at the higher speeds, as will be shown explicitly by the analysis to be given later. The mean effort of the steam on the pistons can be obtained by multiplying the mean area of the low-pressure piston by the reduced mean effective pressure; this effort overcomes the resistance of the ship and also the friction and resistance of the engine

and propeller; these latter are nearly proportional to the resistance; consequently the effort and also the reduced mean effective pressure are nearly proportional to the resistance of the ship, and therefore the curve of reduced pressures can be considered to be a curve of

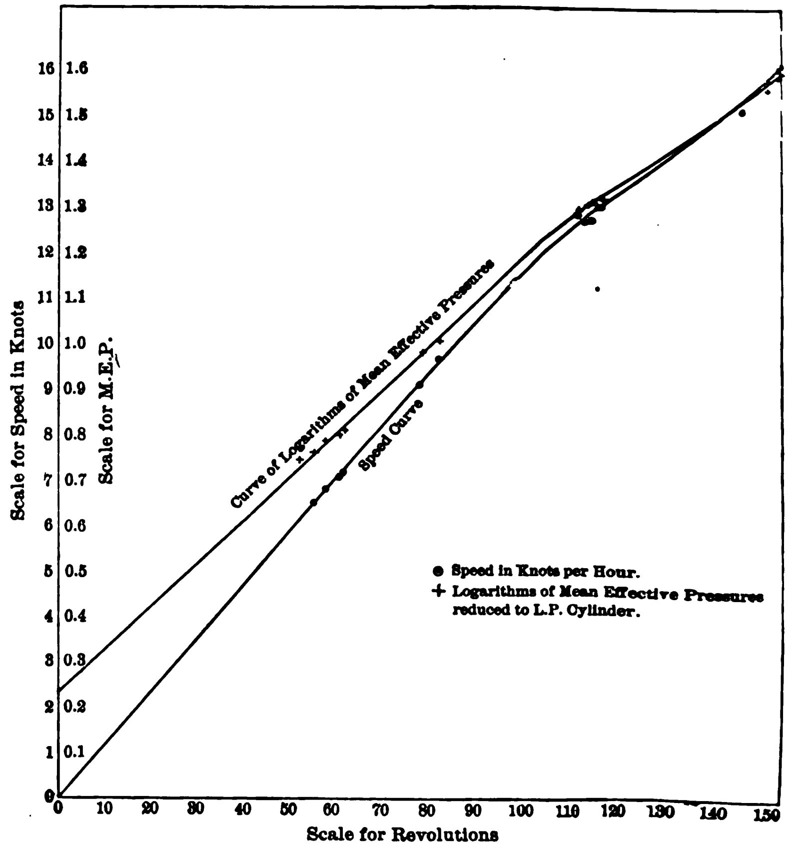

FIG. 205.

resistances. Now the resistance of the ship as represented by equation (1), page 518, contains two terms, the first representing the frictional resistance and the second the wave-making resistance; of these two terms the first varies nearly as the 1.83ths power of the speed, and the second as the 4th power; for moderate speeds the total resistance varies nearly as the square of the speed, but at higher speeds it varies more rapidly. If the resistance varied exactly as the square

of the speed, then the curve of the logarithms of the resistance would be a straight line, and so also would be the curve of reduced pressures; the curve on Fig. 205 has a slight downward curve at the upper end which corresponds to the statement just made; this is a characteristic of resistance curves for a ship which has a good speed for its size; it is clearly brought out by using the logarithms of the resistances or reduced pressures instead of the quantities themselves for the ordinates of the curve. The curve of reduced pressures crosses the axis of ordinates above the origin; in the diagram it crosses at 0.23, which is the logarithm of 1.7; the reduced mean effective pressure corresponding to zero revolutions is called the friction pressure; it varies from 1.5 to 2.5 pounds for modern marine engines.

In the table for the analysis of the progressive speed trials of the *Manning* the speeds are taken at integral knots per hour, from 5 knots to 16 knots, which was the highest speed attained. The corresponding revolutions per minute are directly interpolated from Fig. 205. The reduced mean effective pressure for each speed was also determined from the same figure and used as the basis for the calculation of the indicated horse-power corresponding to that speed. For example, the logarithm of the reduced pressure at 16 knots is 1.583 and the pressure is 38.3 pounds per square inch; the mean area of the low-pressure piston is $2485-\frac{1}{2}\times 16=2477$ square inches; the stroke of the engine is 30 inches, or 2.5 feet; the revolutions per minute are 152; consequently the horse-power for that speed is

$$2477\times 2.5\times 2\times 152\times 38.3\div 33000=2181.$$

It is customary to consider that the power expended in overcoming the friction of the engine, including the friction of the shaft and thrust-block, may be divided into two parts, which are called the initial friction horse-power and the load friction horse-power. The former is considered to be independent of the load and to depend on the initial or friction pressure (in this case 1.7 pounds), and on the revolutions per minute. For example, the initial friction horse-power at 16 knots is

$$2477\times 2.5\times 2\times 152\times 1.7=97.$$

ANALYSIS OF PROGRESSIVE SPEED TRIALS OF U. S. S. MANNING.

1	Speed, knots, V	5	6	7	8	9	[illegible]
2	Revolutions per minute, R	42.8	51.5	60.1	68.8	77.4	[illegible]
3	Indicated horse-power, I.H.P.	69	100	141	194	263	[illegible]
4	Initial friction power	27	33	38	44	49	[illegible]
5	Load friction power	3	5	7	10	15	[illegible]
6	Power delivered to propeller, G.H.P.	39	62	96	140	199	[illegible]
7	Apparent slip, s'	0.041	0.040	0.044	0.044	0.045	[illegible]
8	Slip with 0.15 wake, s	0.184	0.184	0.187	0.186	0.188	[illegible]
9	Propeller efficiency, e	0.655	0.655	0.655	0.655	0.654	[illegible]
10	Thrust horse-power, T.H.P.	30	48	74	108	153	[illegible]
11	Power applied by propeller to propulsion, E.H.P.	25	41	63	92	130	[illegible]
12	Wake gain and thrust deduction	5	7	11	16	23	[illegible]
13	Power to overcome skin resistance, P_s	20	34	52	76	106	142
14	Power to overcome wave-making resistance, P_w	5	7	11	16	24	40
15	Value of constant, b	......	......	......	......	......	......

1	Speed, knots, V	11	12	13	14	15	16
2	Revolutions per minute, R	95.8	106.2	116.7	127.7	139.5	152.0
3	Indicated horse-power, I.H.P.	486	671	920	1345	1661	2181
4	Initial friction power	61	68	74	81	89	97
5	Load friction power	30	42	59	81	110	146
6	Power delivered to propeller, G.H.P.	395	561	787	1083	1462	1938
7	Apparent slip, s'	0.057	0.071	0.085	0.089	0.117	0.135
8	Slip with 0.15 wake, s	0.198	0.211	0.222	0.234	0.250	0.265
9	Propeller efficiency, e	0.654	0.652	0.648	0.644	0.636	0.630
10	Thrust horse-power, T.H.P.	304	431	600	820	1096	1435
11	Power applied by propeller to propulsion, E.H.P.	258	366	510	697	930	1221
12	Wake gain and thrust deduction	46	65	90	127	166	214
13	Power to overcome skin resistance, P_s	187	239	299	369	449	539
14	Power to overcome wave-making resistance, P_w	71	127	211	328	481	682
15	Value of constant, b	......	0.31	0.36	0.38	0.39	0.40

The next step is to estimate the load friction horse-power or the increase of the friction horse-power due to the load on the engine. This is necessarily unsatisfactory because there are no tests on the total friction of large marine engines. It is customary to assume that this increase is 0.07 of the remainder obtained after subtracting the initial friction power from the indicated horse-power. Thus at 16 knots

$$\text{Load friction power} = 0.07(2181 - 97) = 146.$$

The gross horse-power delivered to the propeller is obtained by subtracting from the indicated horse-power the sum of the initial and the load friction powers; at 16 knots the gross horse-power is

$$2181-(97+146)=1938.$$

This gives for the mechanical efficiency of the engine

$$1938 \div 2181=0.88.$$

This is certainly high, but not impossible for a well-built engine in good condition.

It is now necessary to determine the efficiency of the propeller. Three ways appear open to us: (1) to apply Taylor's method; (2) to compare with Barnaby's table, and (3) to assume a probable propeller efficiency. The third method may be used as a check in any case, since we have in general some conception of the probable efficiency of a given propeller. The second method is applicable to blades of the Admiralty pattern; from the known data of the propeller the functions C_A and C_R may be calculated and then the efficiency may be determined by comparison with the table. To apply Taylor's method we may use equation (38), page 461, which gives

$$\text{G.H.P.}=\frac{1}{550\times 60^3}P^3R^3D^2N(asA+fC). \quad . \quad . \quad . \quad (1)$$

Line 6 of the table gives the gross horse-power for each speed; P, R, D, and N represent the known pitch, revolutions, diameter, and number of blades of the propeller; A and C are functions that must be determined from the drawing of the propeller by the method described on page 462; for the propeller in question these functions are

$$A=0.1025, \qquad B=0.1516, \qquad C=0.5305;$$

f is the coefficient of friction for the propeller, which may be assumed to have the value

$$f=0.0160.$$

There now remain two undetermined quantities, s and a, the real slip and the thrust factor.

There being only two unknown quantities, it appears that any two tests at different speeds should give the necessary equations of condition; but such a solution is likely to be unsatisfactory because of the influence of errors of observation, and a better result can be obtained by using the data for all the speeds from 8 to 16 knots, that is, from half to full speed. For this purpose let it first be assumed that the wake factor is 0.10, and let the corresponding values of the thrust factor a be computed by aid of equation (1); the values thus obtained are given in the first line of the following table, and are plotted with the speeds for abscissæ in Fig. 206. Similar computations are made assuming 0.15 and also 0.20 for the wake factor, and the results are set down in the table and plotted on Fig. 206.

WAKE AND THRUST FACTORS.

Speed, Knots.	8	9	10	11	12	13	14	15	16
Thrust factor:									
0.10 wake ...	3.34	3.33	3.30	3.26	3.21	3.13	3.08	2.96	2.93
0.15 " ...	2.50	2.49	2.48	2.49	2.40	2.49	2.49	2.47	2.47
0.20 " ...	1.99	1.97	1.98	1.99	2.03	2.06	2.09	2.06	2.11

The table shows that the assumption of 0.15 for the wake factor gives nearly a constant value for a at all speeds from 8 to 16 knots per hour, and the same fact is even more clearly shown by the figure. The average value of the thrust factor a for the propeller in question appears to be 2.5; this value is appreciably smaller than that given by the equation on page 462,

$$a = 3 - 0.35d = 3 - 0.35\frac{11}{12.33} = 2.7;$$

but the blade is wider than the standard width of $0.2D$ for which the equation was derived, and Durand's experiments as recorded on page 452 show that less thrust is to be expected from wide blades.

The block coefficient for this ship is

$$1000.7 \times 35 \div (188 \times 32 \times 12\tfrac{1}{2}) = 0.47,$$

and the equation for wake coefficient on page 469 gives for this value

$$w=0.20+\tfrac{1}{2}(0.47-0.55)=0.16,$$

which may be considered to be a confirmation of the result derived from the analysis in hand.

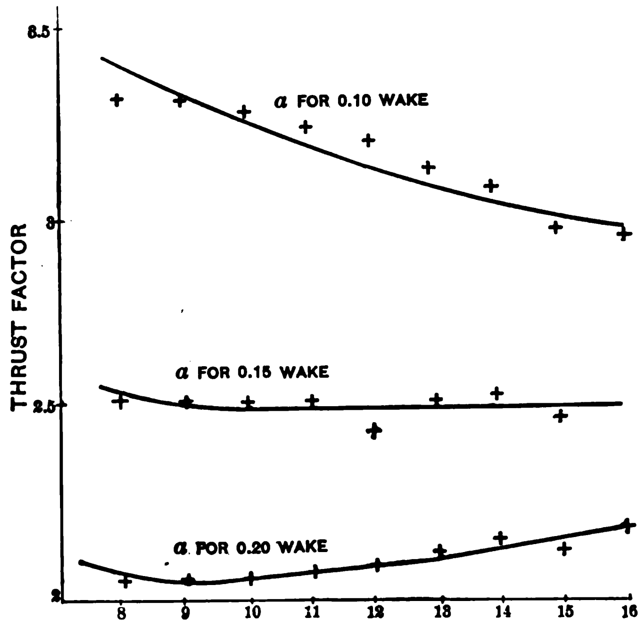

FIG. 206.

It happens that for this analysis one of the assumed wake factors gives nearly a constant value for the thrust factor a; in general this cannot be expected, and it will be convenient to draw a new diagram like Fig. 206a with the assumed wake factors for abscissæ and with values of a taken from the fair curves of Fig. 206 for ordinates; the intersections of the curves of this figure point clearly to the value

$$w=0.15 \quad \text{and} \quad a=2.5,$$

and they will be used for the completion of the table on page 536.

Returning to the table of the analysis of the trials the apparent

the apparent slip is calculated by equation (2), page 470, which may be written

$$s_1 = \frac{PR - V6080 \div 60}{PR}, \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

and the real slip is calculated by equation (6), on the same page,

$$s = w + s_1(1 - w) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

and the results for the several speeds are recorded on the 7th and 8th lines of the table, showing an increase of slip with the speed and especially at the higher speeds.

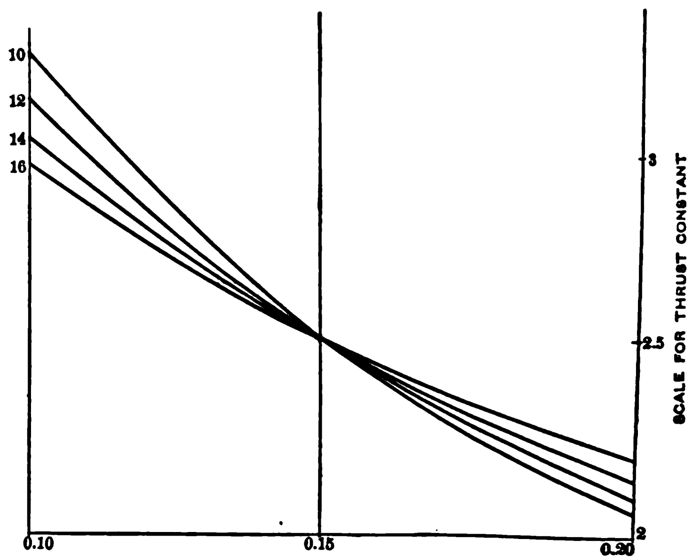

FIG. 206*a*.

Having the wake and thrust factors as well as the functions A, B and C, the propeller efficiency can now be calculated by equation (39), page 461,

$$e = (1 - s)\frac{asA - fB}{asA + fC},$$

after which the effective horse-power can be computed by multiplying the gross horse-power of the 6th line by the efficiency; the horse-power applied to propulsion is set down on the 11th line

The thrust horse-power recorded on the 10th line can be computed by aid of equation (14), page 471, which may be written

$$\text{T.H.P.} = \text{E.H.P.} \div (1 - w). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The difference between the thrust horse-power and the effective horse-power is the wake gain. In general the thrust deduction cannot be determined from the analysis of progressive speed trials; if it be assumed that the hull efficiency is unity, then the thrust deduction is equal to the wake gain, an assumption that is not likely to be much in error for large, well-formed ships.

The effective horse-power which has been derived from the analysis of the speed trials may now be equated to the horse-power as given by equation (3), page 519, under the method of independent analysis:

$$\text{E.H.P.} = 0.00307fSV^{n+1} + 0.003076b\frac{D^{\frac{2}{3}}}{L}V^5. \quad . \quad . \quad . \quad (5)$$

Of the two terms it is considered that the first, which depends on frictional resistance, is best known from experimental investigations; the value of this power to overcome frictional resistance is computed for each speed and set down on the 13th line of the table; subtracting the frictional horse-power just mentioned from the effective horse-power (on the 11th line) gives for the remainder the power to overcome wave-making resistance for each speed, as recorded on the 14th line. The power to overcome wave-making resistance may now be equated to the second term of equation (5), and a value of b may be computed for each speed from 12 to 16 knots per hour; the wave-making resistances at lower speeds is so small that the determination of the factor is unsisfactory.

If the efficiency of the propeller is determined by comparison of the properties with Barnaby's table, then the investigation for determining the wake factor is omitted and the thrust horse-power and wake gain are not determined. There remains in such case only the determination of the effective horse-power (the product of the gross horse-power and the efficiency) and the separation of that net

or effective power into the two parts, one for overcoming friction and the other for overcoming wave-making.

Distribution of Power.—The analysis of the progressive speed trials of the *Manning* affords the means of finding the distribution of the power of the engine. At 16 knots the total indicated horse-power was 2181, distributed as follows:

DISTRIBUTION OF POWER OF THE U. S. S. MANNING.

	16 Knots.		15 Knots.	
Indicated power	2181	1.00	1661	1.00
Initial friction	97	0.05	89	0.05
Load friction	146	0.07	110	0.07
Resistance and friction of propeller	717	0.33	532	0.32
Skin friction	539	0.24	449	0.27
Wave-making	682	0.31	481	0.29

The same results are set down for 15 knots, which is really the proper natural-draught speed of this ship.

The distribution of power for the several speeds is given by Fig. 207, which requires no explanation further than to call attention

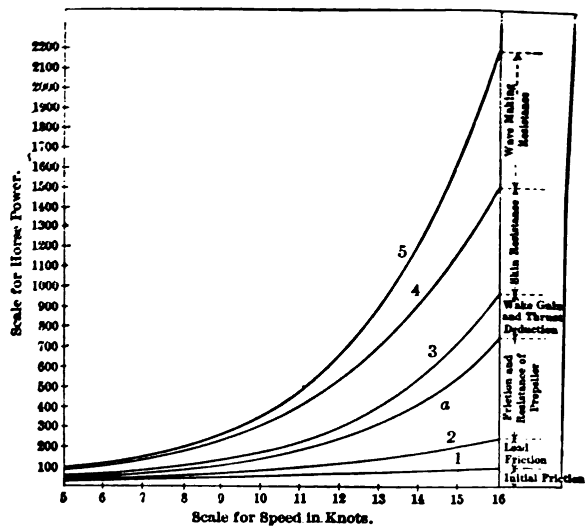

FIG. 207.

to the fact that tne power to overcome the friction and resistance of the propeller is measured between the curves 2 and 3; the wake

gain is measured down from the curve 3 to *a*, and the power lost by the thrust deduction is measured up from the curve *a* to the curve 3; the curve *a* might consequently be omitted from the diagram if desired.

The wave profiles along the hull are shown by Fig. 208 at 16 knots and at 12 knots per hour, which correspond to wave lengths

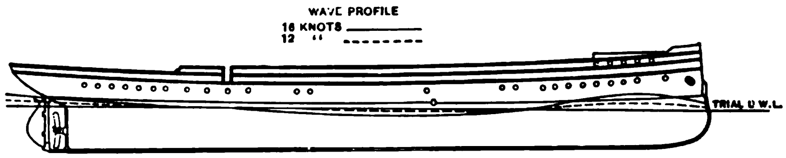

FIG. 208.

of 142 feet and 80 feet respectively, the length of the ship between perpendiculars being 188 feet.

A series of progressive speed trials for the U. S. S. *Yorktown* gave a distribution of power at 16 knots, which is very similar to that of the *Manning* at 15 knots; the full speed of this ship was 16.7 knots.

DISTRIBUTION OF POWER OF THE U. S. S. YORKTOWN.

Indicated power	2952	1.00
Initial friction	146	0.05
Load friction	196	0.07
Resistance and friction of propeller	871	0.29
Skin friction	800	0.27
Wave-making	939	0.32

The following estimate of frictional and other resistances is made by Mr. Blechymden for modern triple-expansion engines:

Initial friction and air-pump	0.076	0.078
Circulating-pump	0.015	.....
Feed-pump	0.006	.....
Bilge-pump	0.005	.....
	0.104	
Load friction 0.075 $(1-0.104)$	0.067	0.069
	0.171	0.147
Delivered to propeller	0.83	0.85

The second column is for ships which have the air-pump only driven from the main engine. Considering that both the *Manning* and the *Yorktown* had independent air-pumps, the engine efficiency found for both (0.88) compares very well with this estimate.

Tidal Corrections.—Progressive speed trials are most commonly made in waters that are affected by tidal currents, for which proper allowance must be made. There are two ways in which this can be done: observers may be stationed near the course who shall make observations on the velocity of the tide, or the ship may be run back and forth over the course with and against the tide. For the first method the observers may be stationed in boats of proper size near the course, and provided with current-meters or other instruments for measuring velocity; since courses will not be laid out where the current is abnormally strong, the tidal velocity is properly a correction and will not call for exceptional refinement in the observations. This method is always used for endurance runs over measured courses, because the tide changes in a marked measure during the time required for the run.

When the ship is run back and forth over the course, the runs at a given speed should be made in immediate succession, and the turns should be made promptly so that there may not be much change in the tide between successive runs. Commonly the arithmetic mean of all the times is taken as the average time.

Sometimes four or six runs in pairs are combined by the method of continued averages, which is easily understood from the following table, in which V_1 and V_2 are the first pair of runs with and against the tide, and V_3 and V_4 are a second pair:

METHOD OF CONTINUED AVERAGES.

Speeds.	First Average.	Second Average.	Third Average.
V_1 V_2 V_3 V_4	$V_a=\frac{1}{2}(V_1+V_2)$ $V_b=\frac{1}{2}(V_2+V_3)$ $V_c=\frac{1}{2}(V_3+V_4)$	$V_\alpha=\frac{1}{2}(V_a+V_b)$ $V_\beta=\frac{1}{2}(V_b+V_c)$	$V=\frac{1}{2}(V_\alpha+V_\beta)$

This method which is equivalent to the operation represented in the following table, is of doubtful value.

Speeds.	First Average.	Second Average.	Third Average.
V_1	$\frac{1}{2}(V_1+V_2)$	$\frac{1}{4}(V_1+2V_2+V_3)$	$\frac{1}{8}(V_1+3V_2+3V_3+V_4)$
V_2	$\frac{1}{2}(V_2+V_3)$	$\frac{1}{4}(V_2+2V_3+V_4)$	
V_3	$\frac{1}{2}(V_3+V_4)$		
V_4			

Under favorable conditions it should be possible to maintain the rate of revolution of the engine for all runs at an intended speed; but if for any reason there is any appreciable variation in the revolutions per minute for a single pair of runs with and against the tide, then the method of taking the mean for the true time on the course is liable to lead to unsatisfactory results when the speed is low. Col. E. A. Stevens* proposes the following method for such cases: Taking C for the length of the course in feet, and F for the flow of the tide in feet per minute, which is supposed to be sensibly constant for a pair of runs, and also t_w and t_a for the times in *minutes*, and R_w and R_a for the number of revolutions per minute, for a pair of runs with and against the tide, he writes for the advance of the ship per revolution

$$\frac{C-Ft_w}{R_w}, \text{ or } \frac{C+Ft_a}{R_a};$$

equating and solving for F, he has for the velocity of the tide in feet per minute

$$F=C\frac{R_a-R_w}{R_at_w+R_wt_a}.$$

Having the velocity of the tide, the distance actually run through the water during the time on the course can be determined for each run just as if special tidal observations were made.

Another method is to plot a diagram like the speed curve of Fig. 205, page 534, for all the runs with the tide, and another diagram for runs against the tide, and to draw a mean curve half-way between in order to find the true curve of speeds and revolutions. This method is especially useful if the tide changes during the trials, the time of the change being indicated by the intersection of the component curves.

* Trans. Soc. Nav. Arch. and Marine Eng.. vol 9.

CHAPTER XIII.

STEERING AND MANŒUVRING.

Two cases arise in the discussion of the steering of a ship, depending on whether it has or has not sails. Steamships have little or no sail and are steered entirely by the rudder; sailing-ships are controlled mainly by the manner in which the sails are set and trimmed, leaving a comparatively small effort to be exerted by the rudder. Steamships have large rudders that are moved by powerful steering-engines, but sailing-ships, even when of large size, are commonly steered by hand and the rudders are much smaller than for steamships.

Resistance to Turning.—When a ship changes its course under the influence of the rudder it turns toward the desired direction and at the same time drifts sidewise away from that direction. Just as for the direct motion of a ship there were found four kinds of resistance, so for both transverse motion and for turning there are four kinds of resistance, namely, stream-line resistance, eddying resistance, wave-making resistance, and frictional resistance; but the relative importance of the several kinds of resistance is quite different. Little is known quantitatively concerning the total resistance to transverse motion or to turning, or concerning the distribution of these resistances among the several components that can be distinguished. It is, however, instructive to consider the matter in a general way and try to determine the relative importance of the components. It is likely that the stream-line resistance to both transverse motion and turning is the most important, and that wave-making resistance, which cannot be dissociated from stream-line resistance, comes next in importance; again, eddy-making resistance is likely to be large, especially at high speeds; finally, frictional resistance has the least importance of any of the four components. These statements are

for well-formed ships, and they may need some modification for very short and full vessels, for which the frictional resistance, especially for slow movements, may have a relatively larger importance.

Thin Plates.—It is customary to refer experiments and discussions of the effects of the rudder to experiments on the resistance of thin plates, and principally those made by Wm. Froude * and by Joëssel †; the former were made in a towing-tank on a small plate, which made a small angle with the direction of motion, principally for the purpose of investigating the action of the screw propeller; the latter were made in an open stream especially for a basis of investigating the action of a rudder.

Froude's experiments can be represented by the equation

$$P_n = 1.7Av^2 \sin i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which P_n is the normal pressure of the water on the surface in pounds, A is the area of the plate in square feet, v is the velocity of the plate in feet per second, and i is the angle which the plate makes with its direction of motion.

Joëssel's experiments were made in the river Loire, near Indret, using a plate which was 0.98 of a foot high, 1.31 of a foot wide, and which had its upper edge immersed 0.66 of a foot below the surface of the water. Experiments were made with varying velocities of current, the maximum being 4.3 feet per second, or 2.5 knots per hour.

Two different series of tests were made, the first to determine the point of application of the normal pressure on the plate, and the second to determine the amount of the pressure. For the first series the plate was hung freely on a vertical axis that could be placed at any distance from the leading edge of the plate. When the plate was exposed to the current of the river it took an angle depending on the position of the vertical axis. When the axis was at the middle of the plate it stood at right angles with the current, and as the axis was moved toward one edge the plate took a continually smaller angle with the stream. The angle approached zero when

* Proc. Inst. Nav. Archts., vol. xix.

† Mémorial du Génie maritime, 1873.

the axis approached one-fifth of the width of the plate from the leading edge. Joëssel deduced from these experiments the following equation:

$$x_i = l(0.195 + 0.305 \sin i). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

in which x_i is the distance of the axis (or of the point of application of the resultant pressure) from the leading edge corresponding to the angle i between the plate and the direction of motion of the water; l is the width in feet.

In the second series of tests the plate was hung on a vertical axis at one edge, and was held at various angles with the current. The moment of the pressure tending to turn the plate about the axis at the leading edge was measured by a cord passing around a pulley on the axis and then over a pulley on a horizontal axis, finally the cord was loaded by the proper weight in a scale-pan. The moment was found to be represented by the equation

$$M_i = 0.787Alv^2 \sin i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which M_i is the moment expressed in terms of the foot and pound, A is the area in square feet, l is the width in feet, and v is the velocity in feet per second. But the moment is equal to the normal force acting on the plane, multiplied by the distance of the point of application from the leading edge, so that if the normal force is P_n, then

$$P_n x_i = M_i.$$

Combining this conclusion with equations (2) and (3)

$$P_n = \frac{0.787Av^2 \sin i}{0.195 + 0.305 \sin i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the plate is at right angles with the stream, the normal pressure becomes

$$P_{90} = 2 \times 0.787Av^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that

$$P_n = \frac{1}{2} P_{90} \cdot \frac{\sin i}{0.195 + 0.305 \sin i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is convenient to introduce the density of the liquid, 62.5 pounds per cubic foot, and the acceleration due to gravity, 32.16 feet per second, which gives in place of equation (4)

$$P_n = 0.405 \frac{w}{g} A v^2 \frac{\sin i}{0.195 + 0.305 \sin i}. \quad . \quad . \quad . \quad . \quad (7)$$

For angles not exceeding 50° Joëssel proposes the equation

$$P_n = 2.13 A v^2 \sin i = 1.10 \frac{w}{g} A v^2 \sin i, \quad . \quad . \quad . \quad . \quad (8)$$

and for angles not exceeding 15° the equation

$$P_n = 2.85 A v^2 \sin i = 1.47 A v^2 \sin i.$$

For large planes it is recommended that the numerical constants be reduced by about one-tenth of their assigned values.

Towing.—The experiments on thin planes suggest an explanation of certain phenomena which have been observed in towing ships. If a ship is towed from a point near the bow it will steadily follow the course of the tow-rope. But if the point of attachment of the tow-line is carried back from the bow it will tow unsteadily and will tend first to yaw from side to side, and, if the point is far enough back, to tow at an angle with the course. It has been found that for thin plates the axis of support must be less than $\frac{1}{8}$ of the width of the plate from the forward edge, if it is desired that it shall remain parallel to the stream in which it is immersed. The corresponding point for a body shaped like a ship is not well determined; it is safe to assume it to be less than $\frac{1}{8}$ of the length of the ship. Conversely, if a ship is towing another ship, the tow-line must not be attached at a point too far forward, otherwise the former will become unsteady and will steer with some difficulty. A ship at anchor in a stream is in the same condition as a ship which is towed in quiet water. Some French naval vessels have the bow drawn back above the water-line to reduce the weight forward and thus avoid heavy pitching; they have been found to behave in an unsatisfactory manner when riding at anchor. Of course a ship when towed or when lying at anchor in a stream will mind the helm as though propelled by steam.

Forms of Rudders.—Large sailing vessels commonly have the rudder shaped like Fig. 209, *A*, and the area of the rudder is comparatively small, for the sail is so set as to balance and leave comparatively little for the rudder to do, and the ship is steered by hand so that a large rudder cannot be conveniently controlled. Yachts usually have the rudder shaped like Fig. 209, *B*, and the area is relatively larger, as the yacht must manœuvre rapidly. There does not seem to be any good reason for the difference in shape. The rudder-post for ships is vertical or slightly inclined; for yachts it has a considerable rake. The rake of the rudder-post of a yacht is

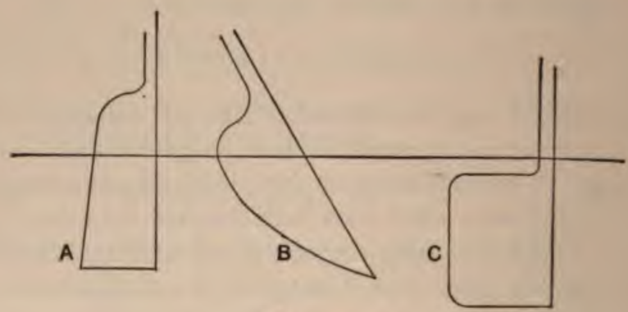

FIG. 209.

determined in part by the general form of the hull and in part by certain racing rules. If a ship is propelled by steam the rudder-post usually is vertical, and clearly ought to be so, as otherwise when the rudder-post has a rake the effective angle of the rudder will be less than the angle of helm and there is a downward component tending to depress the stern.

A yacht which has the sails properly balanced will always come up to the wind when the helm is abandoned, consequently when sailing on the wind the rudder is turned away from the wind and the yacht is inclined to the leeward. If the rudder-post were vertical when the yacht is erect, the effective angle of the rudder when inclined would be less than the angle of helm and there would be an upward component tending to raise the stern. The rudder-post has a rake and this tends to neutralize the effect of the inclination. Naval vessels usually have the rudder rectangular, like Fig. 209, *C*,

and wholly immersed. Merchant steamers have the rudder more or less rectangular, but they are not wholly immersed.

Compensated Rudders.—The experiments of Joëssel show that a plane will take an angle with the stream when immersed, unless the axis is less than $\frac{1}{5}$ of the width from the forward edge. The conclusion from this appears to be that a rectangular balanced rudder should be hung at $\frac{1}{5}$ or less of its width from the forward edge. Experiments with rudders show that the axis may be slightly further aft. This rule is sometimes extended to rudders which are not rectangular in shape by making the moment of the area forward of the rudder-post one-fourth of the moment of the area abaft the rudder-post.

Size of Rudders.—The proper size to be given to rudders is determined entirely from experience. The following table gives ordinary proportions expressed in the form of the ratio of the area of the rudder to the area of the lateral plane of the ship.

SIZE OF RUDDERS.

Ratio of area of rudder to area of lateral plane.

	Unbalanced Rudders.	Compensated Rudders.
Paddle-wheel boats	0.021	
Large passenger ships	0.016	
Ordinary screw ships	0.020	0.024
Armored ships	0.025	0.030

In general it is desirable to make rudders deep and narrow, as for the same area and turning effect there is smaller twisting moment on the rudder-head. Naval vessels, however, must have the rudder and steering gear below the water-line and consequently have wider rudders.

Size of Rudder-head.—The size of the rudder-head of a merchant steamer is habitually taken, togetherwith the general scantling for the ship, from tables prepared either by some governmental authority or by boards of insurance underwriters. Such tables usually do not take explicit cognizance of the speed of the ship, which is not altogether illogical, since the greatest stresses in the rudder-head may come from blows of the sea. The rudder-head for a warship is determined by comparison with former practice.

Though Joëssel's tests were made on a small plate and with low velocities, they may be taken in conjunction with some experiments on rudders as the basis of an equation for calculating the stress in the rudder-head. In the first place it is found that the greatest turning moment of a rudder is found for a moderate angle of helm, and many ships have stops limiting the angle to 45°. And further it is found that the moment transmitted to the rudder-head is less than that computed by Joëssel's formula for considerable angles of helm and for good speeds; unfortunately there are not tests enough to enable us to formulate a general relation. If the angle in equation (3), page 548, is made 45°, then as a rough approximation the moment in foot-pounds may be written

$$M_{45} = 0.55 A l v^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the area is taken in square feet and the width in feet. The moment of resistance to twisting of the rudder in inch-pounds may be written

$$M_t = \sigma \frac{I}{y} - o \frac{\frac{\pi}{32} d^4}{\frac{1}{2} d} = \sigma \frac{\pi}{16} d^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where I is the polar moment of inertia of the section of the rudder-head, y is the most strained fibre, and d is the diameter in inches; σ is the stress in pounds per square inch. In order that these two moments may be equated they must be reduced to the same terms, which can be conveniently done by multiplying equation (1) by 12. After reduction the equation for diameter may be written

$$d = \sqrt[3]{33 \frac{A l v^2}{\sigma}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which A is the area of the rudder in square feet, l is the maximum width, v is the velocity in feet per second, and σ is the working stress in pounds per square inch; d is the diameter of the rudder-head in inches. The working stress should be low to allow for unknown effects of the action of the sea, 6000 or less. If preferred, equation (3) may be expressed as a proportion for basing the size for a new ship on that of a ship in service; thus,

$$d_1 : d_2 :: \sqrt[3]{A_1 l_1 v_1^2} : \sqrt[3]{A_2 l_2 v_2^2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

General Discussion of Steering.—It does not appear to be possible at present to deduce a general equation to represent the effects of the rudder on a ship, for there is not sufficient experimental basis to determine the forms of the several elements of such an equation. And yet the statement of the problem gives a valuable insight into some of the effects of the rudder on the ship, and, conversely, of the ship on the rudder.

Suppose that a ship is moving in a straight line, and that the helm is suddenly put over to the angle α, as shown by Fig. 210. Then there will be developed the several forces shown on that diagram; the forces are not drawn to the same scale. and the diagram gives only relative results. The water striking on the front of the rudder gives rise to the normal force P_n, which may be resolved into two components, one parallel to the axis of the ship, acting as additional resistance, and the other at right angles tending to move the ship bodily toward the right. The water running past the rudder gives rise to the frictional resistance P_s, which may also be resolved into two components, one parallel to the axis, and one at right angles, the latter tending to move the ship to the left. The difference of the transverse forces forms a resultant which gives rise to an equal and opposite resistance R_t at the centre of lateral resistance of the ship. There is, therefore, a couple tending to turn the ship around a vertical axis at the centre of gravity, which may for present purposes be assumed to be at the middle of the length. The two forces parallel to the axis of the ship form an additional resistance, to be added to the ordinary direct resistance R_d. The resultant direct resistance may be compounded with the transverse resistance R_t, giving the total resistance R. This resistance

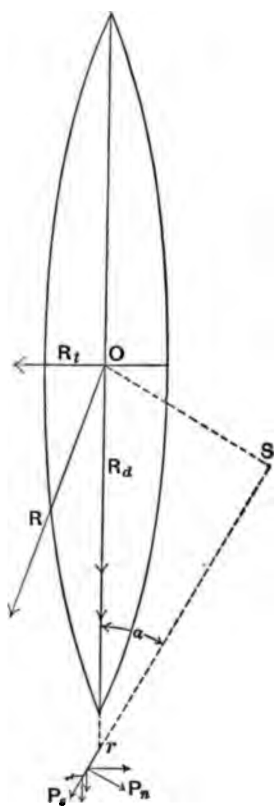

FIG. 210.

is considerably in excess of the ordinary direct resistance, which is just overcome by the propulsive force; the deficit must be made good by the stored energy in the ship, and the ship consequently loses speed when the helm is put over.

The effect of the forces acting on the ship when the helm is put over is to cause it to move away from the direction in which it is desired to go, and to swing round an axis at about the middle of its length in the proper direction; the forward motion of the ship carries it along a curved path in the direction desired.

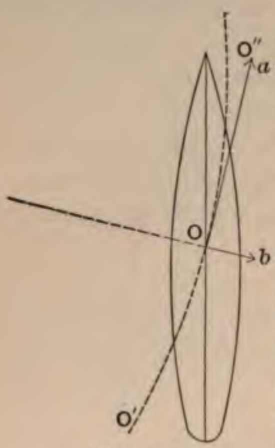

FIG. 211.

When the ship has begun to swing around under the influence of the rudder, it moves along a curved path as indicated by $O'OO''$ in Fig. 211, and it lies with the bow turned toward the inside of the curve. For the ship's resultant velocity at any instant (represented by Oa) is made up of a longitudinal and a transverse component, and the path of the centre of the ship will have Oa for a tangent. The tangential velocity Oa is accompanied by a radial acceleration Ob, which adds to the lateral drift of the ship.

The effect of friction on the rudder is a resistance or drag and tends to check the speed of the ship and to hinder the rotation or swinging; fortunately it is small compared with the other forces, and may be neglected in the remainder of the discussion.

Phases of Steering.—In the preceding discussion the helm was considered to be put over suddenly; even with powerful steam steering-engines some time is taken for this manœuvre. The time required for this is called the phase, or *period of manœuvring*. During this time the ship acquires considerable motion, both of drift and of swinging; the consequence is that the water does not strike the rudder at the angle of the tiller α, but at a less angle $\alpha - \beta$, and the ship also loses some headway.

After the helm is put over to a certain angle the ship continues to turn more and more quickly, until it finally proceeds at uniform

speed in a circle. There are then two other phases or periods, the period of ***variable motion*** and the period of ***uniform motion.***

The period of manœuvring for ships that are steered by power is always short. It is interesting since the greatest moments on the rudder-head are then experienced.

The period of variable motion is really of the greatest practical importance, since it extends over a time covering the turning of the ship through 90° to 180° if not more. A ship seldom changes course by an angle so large as a right angle, especially when at high speed.

The period of uniform motion in a circle is most easily and therefore most often studied; its study is important, since the diameter of the circle of uniform motion is a fair indication of the general manœuvring power of the ship.

Coefficient of Reduction.—There are two objects to be considered in the discussion of steering: the effect on the rudder, especially as affecting the moment on the head of the rudder tending to twist it, and the behavior of the ship.

The investigation of the effect on the rudder may be based on Joëssel's experiments on a thin plate immersed in a steady stream, as represented by equation (3), page 548.

$$M_i = 0.787Alv^2 \sin i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To apply this to a rudder of a ship which has begun to swing, the angle i may be replaced by the angle $\alpha - \beta$ at which the water strikes the rudder; α may be taken as the angle of the helm, and β as an unknown small angle which increases with α. The velocity may be replaced by the velocity v_1 of the ship in feet per second, which is always appreciably less than v_0, the velocity on the course before the helm was put over. It is customary to compare the moment thus obtained with a fictitious moment computed with the angle of the helm α and the speed of the ship on the course. This ratio may be written

$$\frac{V_1^2 \sin(\alpha - \beta)}{V_0^2 \sin \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

using V_0 and V_1 to represent the speeds of the ship in knots, before and after the helm is put over. This ratio is called the *coefficient*

of reduction and is used to allow for the two unknown elements, namely, the reduction of the angle of inclination and the reduction of the velocity of the ship. The coefficient of reduction should be determined from tests on various types of ships, but such tests are not easily made for ships that have the common forms of steam steering-engines, and our knowledge of the coefficient is not sufficient to give general rules for computing the stress in rudder-heads.

The reports of tests to determine the turning moment of the rudder give the speed of the ship on the course before the helm was put over, the angle of the helm, and the coefficient of reduction which is assumed to be the ratio of the actual moment on the rudder-head divided by the fictitious moment computed by Joëssel's equation using the angle of the helm for i, and the speed of the ship on the course for v. The rate at which the helm is put over is usually reported also. The following tables give the results of tests on the French torpedo-gunboats *Condor* and *Épervier*, which were 223 feet long and had a displacement of 1270 tons.

EXPERIMENTS ON THE RUDDER OF THE CONDOR AND ÉPERVIER.

Initial Velocity Knots.	Angle of Helm. (The helm was put over at rate of 3° per second.)		
	9°	18°	28°
6.......	0.792	0.723	0.704
8.......	0.730	0.650	0.630
10.......	0.675	0.584	0.570
12.......	0.620	0.528	0.520
14.......	0.565	0.480	0.478
16.......	0.515	0.443	0.443
18.......	0.465	0.414	0.412
20.......	0.420	0.389	0.385

The most extensive experiments appear to be those by Naval Constructor Elliot Snow, U.S.N.,* on the monitor *Monterey*. This ship has a hydraulic steering-gear, so that the moments acting on the rudder were inferred from the reading of gauges which showed the hydraulic pressure acting on the gear. Preliminary experiments

* Trans. Soc. Nav. Arch. and Marine Eng., vol. 3.

were made to determine the friction of the hydraulic gear, but there of course remains the unavoidable uncertainty due to the method. It is to be regretted also that the tests were made on a special type of vessel, and that the speeds were relatively low. The principal dimensions of the ship are:

Length	256 ft.
Beam	59 ft.
Draft	14 ft. 6 ins.
Displacement	4027 tons.
Area of rudder	95.7 sq. ft.
Width of rudder	10.4 ft.
Ratio of area of rudder to area of lateral plane	0.0261

The results are given in the following table:

RUDDER TESTS ON THE U. S. S. MONTEREY.

Speed, Knots.	Angle of Helm.				
	15°	20°	25°	30°	35°
4	1.102	0.900	0.811	0.730	0.649
6	0.501	0.525	0.519	0.491	0.459
8	0.321	0.378	0.394	0.389	0.370
10	0.259	0.327	0.354	0.361	0.343
12	0.250	0 322	0.358	0.363	0.353

When tests are made to determine the moment of the water tending to turn the rudder, it is found that the moment increases as the helm is moved over till the helm reaches the desired angle and is stopped; then the moment experiences a sudden and sharp decrease. The explanation is that when the rudder is moved the moment exerted to turn it is the sum of the moments required to of the rudder pressure of the water plus the moment of the friction overcome the rudder in its bearings; when the rudder is stopped, the moment becomes the difference of the preceding moments, with perhaps some change due to excess of static over moving friction. It appears that the greatest moment is the one having the real interest, as it is the moment that must be endured by the rudder-

head. It is, of course, proper to measure both so as to attempt to allow for friction, which may vary from time to time and from vessel to vessel.

Not enough has been learned by such experiments to serve as a true guide, but the provisional conclusion is that the largest moment that will be experienced at highest speed will be materially less than that calculated by Joëssel's equation, and that rudder-heads may be designed with that as a basis will be safe, even if it be considered that rudders experience severe shocks from blows of the sea.

Effect of Propeller on Rudder.—The rudder of a single-screw ship is habitually placed directly aft of the propeller, and is, of course, affected by the race from the propeller, both on account of the sternward velocity of the race and on account of its rotation. The velocity of the race is a direct advantage and may enable the ship to steer even when it has not yet acquired headway, provided, of course, that the propeller is driven forward. A ship may be made to swing around by first backing and then going ahead with the rudder thrown over. On the other hand, the drag of the rudder due to friction is greater on account of the race; but this is probably not important.

The rotation of the race of the propeller affects the neutral position of the rudder and the facility of turning to the port or starboard. A right-handed screw immediately in front of the rudder throws a stream of water on the rudder which is inclined to the starboard near the surface and to the port near the keel; the water near the keel is less broken and has the greater effect so that the neutral position of the rudder is inclined to the port, especially if it is balanced; the helm is of course inclined to the starboard. It is recommended by Pollard and Dudebout that the rudder be twisted into a helicoidal form to counteract this effect. On the other hand Rankine points out that a plain rudder has a beneficial effect in straightening out the race to a certain extent, and he suggests that the rudder be twisted so that the water striking it may have a forward component and that some energy may be thus recovered from the race. It is improbable that any appreciable effect can be obtained from this device.

Experience shows that a rudder placed in front of a propeller has much less effect than when placed behind it. In the first place the rudder does not there get the benefit from the race of the propeller, and in the second place the rudder directs an oblique stream on to the screw, under the influence of which the screw tends to turn the stern in the wrong direction. Suppose that the propeller is a right-handed screw and that the ship is turning to the starboard, the rudder being inclined in that direction; the rudder directs an oblique stream across the propeller from port to starboard which a blade near the surface of the water will cut more nearly at right angles and will have less effect than usual either for propulsion or for transverse motion; meanwhile the blade which is near the keel will cut the stream at a smaller angle and will act somewhat like a steering oar to throw the stern to the starboard. If the ship turns to the port, the propeller blade which is near the surface will have the preponderating effect which resists the proper action of the rudder, only in this case the blade near the surface acts on broken water, and consequently its effect is less marked. The ship will consequently turn more readily to port than to starboard with a right-handed screw.

Mr. William Froude showed by some experiments on models that there is a gain in efficiency by placing the propeller far aft of the stern, a quarter or more of the length of the ship. Such an arrangement would naturally have the rudder in front of the propeller near the stern, where it would have little or no effect on the propeller; but there are very apparent objections to such a disposition of the propeller in practice.

Bow Rudders are seldom used except on ferry-boats or other ships that steam in either direction. On such ships the bow rudder is commonly fixed in a central position, and the stern rudder only is depended on for steering. The water flowing past the stern of a ship is directed by its form on to the rudder in a favorable manner, and when the rudder is turned it diverts the water and there is a strong pressure exerted on the rudder to throw the stern of the ship around. When a bow rudder is turned to one side, the starboard, for example, an eddy forms on that side (starboard) of the rudder and there is some divergence of the water past the bows, especially on the con-

trary (port) side, but the entire effect is feeble compared with the effect of a stern rudder.

Effect of Rudder when Backing.—When a paddle-wheel steamer backs the rudder turns the ship in the same manner as when going ahead, but it is much less efficient, acting much like a bow rudder.

The rudder of a screw ship when backing has very little effect, and what effect it has is quite uncertain. The rudder does not have the effect of the race from the propeller, and it throws on to the propeller an inclined stream which makes the propeller tend to throw the stern the wrong way; at the same time the rudder has less action of its own, as is the case with the paddle-wheel steamer.

Twin Screws.—If a ship has twin screws they may be run at different speeds, or one may be reversed to aid the rudder, or the ship may be steered by the screws only. For efficient action of twin screws for steering, the shafts should be parallel or convergent astern.

Inclination when Turning.—When a ship is on a circular course, as in the period of uniform motion, or indeed on any curved path, there will be acting on it a centrifugal force applied at the centre of gravity, which is commonly near the water-line, and also a transverse force at the centre of pressure of the rudder.

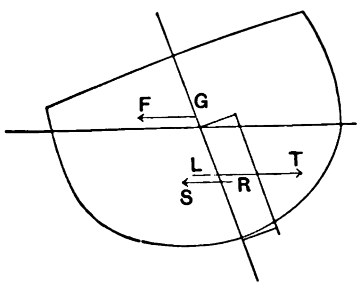

FIG. 212.

Both forces act in the same direction and are resisted by the transverse resistance of the ship applied at the centre of lateral resistance.

Thus in Fig. 212 the centrifugal force FG is supposed to be turned away from the centre of curvature of the ship's path; LT is the lateral resistance turned toward the centre, and RS is the horizontal component of the force on the rudder turned away from the centre. The ship will commonly be inclined away from the centre of curvature of the path since R is commonly somewhere near the same height as the centre of lateral resistance L. If R is below L it has a tendency to

incline the ship toward the centre of curvature of the path, and if its influence is sufficient it may correct the tendency of the centrifugal force to heel the ship away from that centre. Some torpedo-boats which have small draught have been given deeply immersed rudders to correct the tendency to heel over when turning at high speeds. In the period of manœuvring, while the rudder is moving over, but the ship has scarcely begun to change her course, the pressure on the rudder is considerable but the centrifugal force is small; the ship will then heel toward the direction in which it is turning. Conversely, if the rudder is moved quickly to the middle position while the boat is turning sharply, the righting effort of the rudder is lost and the boat may heel over to a dangerous degree. The coxswain of such a boat must be cautioned to have this in mind when he receives the order to steady the helm.

Path of Ship when Turning.—Suppose a ship to be on the course *am* when the helm is put over to a definite angle α, and that the engine is allowed to run under constant steam pressure, it will swing around more and more and will at the same time lose speed, and the engine will turn more slowly till she finally goes round in a circle of fixed diameter, as shown by Fig. 213. Successive positions of the bow are shown by b, b', b'', and b''' and of the stern by s, s', s'' and s''', while the path of the middle point of the ship is shown by a, a', a'', and a'''. It is to be noted that the ship first moves bodily to the left of the line *am*, and that it is carried over that line to the right by proceeding along the curved path a, a', a'', and a''', as it continually changes the course. The bow moves gradually to the right from the instant that the helm is put over, but the stern makes a wide sweep to the left of the line *am*. The diameter of the turning circle in Fig. 213 has been chosen very small compared with the length of the ship, in order to bring out the features of the movement of the ship. It is usually three or four times the length of the ship, and the path over which the ship sweeps is much narrower.

The motion of the ship while making a turn like that represented by the figure consists of a forward motion, a swinging about an axis at the centre of lateral resistance (which is near the middle of the length), and a transverse drift. There is a point somewhere between the middle and the bow at which the lateral drift away from the

centre of the path is equal to the motion due to swinging abo[illegible]
centre of lateral resistance, and consequently a person stand[illegible]
this point will not perceive any transverse drift. This point [illegible]
a little as the radius of the path changes, but not very far. It [illegible]

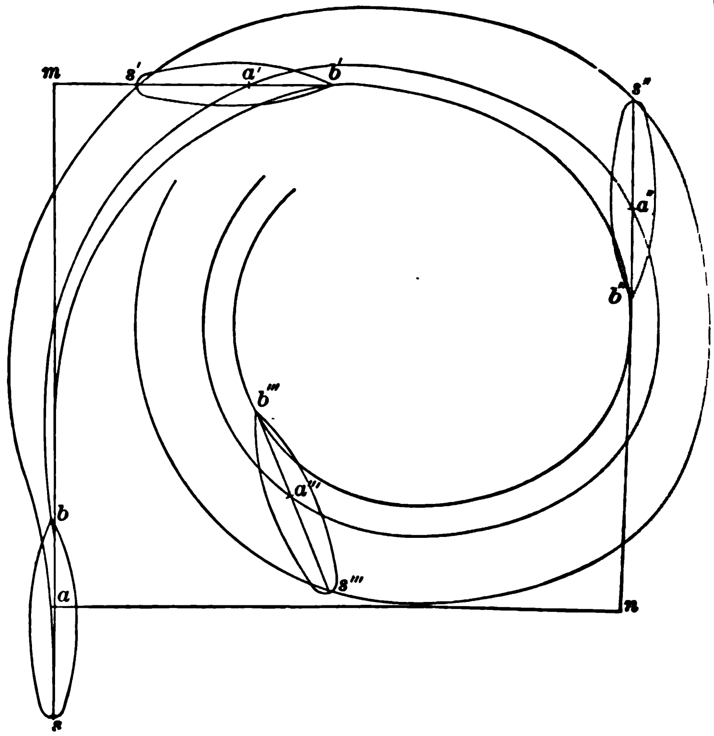

FIG. 213.

be called the *point of gyration*, and appears to be a proper location f[illegible]
the pilot-house. Modern torpedo-boats have deep V-shaped sectio[illegible]
forward and flat U-shaped sections aft and the stern cons[illegible]
quently swings easily; such boats turn about an axis forward of t[illegible]
centre of lateral resistance, which is forward of the midship sectio[illegible]
and have the point of gyration very near the bow.

It is evident that to a person standing near the point of gyrati[illegible]
the bow will appear to swing only a moderate amount to the l[illegible]

of the path of that point, in Fig. 213, but that the stern will appear to swing far outside of that point. A proper understanding of this matter is essential to the pilot, or he may avoid an obstacle with the bow but strike it with the stern. The pilot should avoid placing his ship close alongside another ship when both are going in the same direction, for a collision by the stern is certain to follow if either swerves from its course. In fact, if one ship in such a position turns in a given direction, the other must turn in the same direction to avoid a collision. The safest way of separating is for one ship to draw ahead, or the distance between the ships may be gradually increased until a turn can be made safely. Much the same condition arises when a ship is brought alongside a pier by making a long turn, for, though the ship can be checked by reversing the engine, it will continue to swing, and, if proper allowance has not been made, the stern may strike the pier.

In the discussion of the manœuvring properties of a ship attention is given to (1) the distance $am = L_1$ that the ship will move in the direction of the former course by the time that it has made a turn to right angles or eight points with that course; (2) the distance an between the former course and a course $a''n = L_2$ in the opposite direction; and (3) the diameter D of the turning circle. These features should be determined during the turning tests of a ship; often the diameter of the turning circle only is determined.

Manœuvring Properties.—While a useful analytical discussion of the manœuvring properties of a ship does not now appear to be possible, the following general statements will be found reasonable and in conformity with experience.

In addition to the three properties mentioned in the preceding section, there may be distinguished two other properties: (4) the angle δ which the ship's axis makes with the path of the point of gyration, and (5) the reduction of velocity due to changing course, which may be expressed by the ratio $V_0 \div V_1$, as on page 555 in the discussion of the coefficient of reduction.

These several properties are evidently related to each other, and it would appear that an adequate theory of manœuvring, with proper experimental constants, should enable us to infer one property from another; but this can now be done only in a

very vague manner. The following general relations will be found to exist:

a. The angle δ increases and the diameter D decreases as the angle of the helm increases, but not proportionally to that angle. In fact the maximum manœuvring power is attained with very moderate angles of helm, 35° to 45°.

b. The angle δ increases and the diameter D decreases as the ratio $G \div A$ of the area of the rudder to the immersed area of the shear-plan increases.

c. Again, δ increases and D decreases as the ship is made fuller, because the rounded outlines of the water-lines offer less resistance to turning.

d. If a ship is increased in draught, the area of the rudder (or its depth) should be increased in like proportion to give the same manœuvring power.

e. The speed V_1 when turning, or the ratio $V_1 \div V_0$, decreases with the angle of helm and with the area of the rudder. On the other hand it falls off less rapidly for a full than for a fine ship.

It is probable that the manœuvring properties of similar ships sollow the general laws of similitude, so that the several dimension L_1, L_2, and D are proportional to the lengths. Again, the turning moment of the rudder is proportional in some manner to the moment of its area about a vertical axis through the centre of lateral resistance of the ship, and the moment of resistance to turning for similar ships is likewise proportional to the area of the shear-plan below the water-line. Consequently the areas of rudders should vary as the square of a linear dimension, or as the immersed area of the shear-plan.

Experiments show that the diameter of the circle of uniform motion decreases a little as the speed of the ship increases; that is, a ship manœuvres a little better at high speed. The difference is unimportant.

Experimental Results.—The following results of experiments on the French ship *Terrible* are given by Pollard * and Dudebout:

* Théorie du Navire, vol. 4, p. 82.

TURNING TESTS ON THE TERRIBLE.

Angle of the helm, degrees......	34	34	34	20	20	20
Revolutions per minute before turning..................	80	60	40	80	60	40
Diameter of turning circle of rudder-head, metres:						
Starboard............	402	444	436	548	544	543
Port................	447	444	436	514	525	543
Speed in knots of rudder-head:						
Starboard............	9.8	7.9	5.3	11.4	8.2	5.5
Port................	10.8	7.9	5.3	11.3	8.3	5.5
Time for entire turn, 360°, minutes and seconds:						
Starboard............	4–21	6–3	9–17	5–3	7–9	10–20
Port................	4–22	6–3	9–17	4–55	6–41	10–20

The table on p. 566 of the manœuvring properties of several French naval vessels are taken from Pollard and Dudebout; the co-efficient is the result obtained by taking the cube root of the displacement and dividing by the area of the midship section and multiplying by the length.

Determination of Manœuvring Properties.—The complete determination of the manœuvring properties of a ship requires the location of the ship both as to position and direction, as in Fig. 213, page 562, beginning at the time when the ship is following a determined course at a known speed and ending when the ship is circling round on its turning circle at a uniform speed. More frequently the investigation is limited to the determination of the diameter of the turning circle and the time required for a complete turn. In either case the observations are likely to be made on board, and will require four observers, who may be assisted by one or more recorders. One observer may note the time and give the signal for the other observers, who must take simultaneous observations; a second observer will determine the ship's course, preferably by measuring the angle between the fore-and-aft axis of the ship and a line to some distant object, usually the sun; the other two observers will be placed at the ends of a base-line on the ship, one near the bow and the other near the stern, and will determine each the direction from his post to a fixed point at a moderate distance. The observations at the two ends of the base-line will determine a triangle with the fixed point at its apex, and thus determine the ship's distance from the

NAUTICAL PROPERTIES OF SHIPS.

Name of Ship.	Coefficient, $\frac{D^{\frac{1}{3}}}{B}L$.	Ratio of Area of Rudder to Area of Sheer-plan.	Length, Meters.	Angle of Helm. Deg.	Initial Velocity, Knots.	Time for Complete Turn.	Reduction of Velocity, $V_1 \div V_0$.	Ratio of Diameter of Turning Circle to Length.
Duquesne......	22.95	0.0235	101.30	34	10.5	7–31	0.738	2.78
				20	10.1	10–33	0.931	5.00
Trident.......	17.26	0.0250	102.00	35	13.0	6–2	0.615	2.18
				35	10.0	7–32	0.750	2.72
				35	7.5	10–46	0.786	3.04
				20	13.0	6–25	0.769	3.09
				20	10.0	8–50	0.850	3.81
				20	7.2	12–45	0.857	4.10
Flamme.......	18.29	0.0440	50.55	35	11.2	3–20	0.851	1.87
				35	6.7	3–40	0.843	1.70
				15	11.1	5–00	0.925	3.69
				15	6.6	5–50	0.912	3.40
Friedland......	15.97	0.0237	98.71	34	12.0	6–50	0.583	2.49
Montcalm.....	15.59	0.0260	73.52	33	10.6	5–10	0.583	2.45
				33	7.1	7–27	0.583	2.05
				25	10.6	5–22	0.583	2.51
				25	7.1	7–26	0.583	2.23
Terrible.......	15.23	0.0245	88.34	34	13.2	4–21.5	0.779	2.40
				34	10.9	6–03	0.722	2.51
				34	7.4	9–17	0.717	2.47
				20	13.2	4–59	0.861	3.00
				20	10.9	6–55	0.757	3.02
				20	7.4	10–20	0.740	3.07
Indomptable...	15.23	0.0245	88.34	35	13.9	5–55	0.743	2.57
				35	12.0	6–55	0.678	2.47
				35	9.2	8–10	0.730	2.77
				20	13.9	6–05	0.811	3.13
				20	12.1	7–10	0.741	3.33
				20	9.2	8–50	0.739	3.17
Tempête......	15.09	0.0360	78.60	33	11.75	4–00	0.430	1.13
				33	9.70	4–35.5	0.495	1.10
				25	11.20	4–16	0.500	1.23
				25	9.725	4–39	0.575	1.37
				10	11.00	5–10	0.640	2.54
				10	9.85	5–22	0.710	2.24
Admiral Baudin	15.07	0.0251	104.40	34	13.35	5–22	0.783	2.43
				20	11.6	5–55	0.796	2.61

point; the observation of the course by the sun will determine the direction of the ship from the fixed point; the observations of time show the times required to pass through the several phases of evolution. The requisite measurements of angles may be made with sextants, or compasses, or plane-tables. The point from which the distances are measured may be conveniently a pole buoy ballasted to float erect, placed so that the ship will circle round it when on its turning circle. Observations on the sun require an allowance for the apparent motion of the sun by methods familiar to a navigator. A distant terrestrial object may be substituted provided that it is some well-defined landmark at a sufficient distance.

The diameter of the turning circle, especially of a small boat, may be determined by taking a photograph of the boat and its circular wake when the boat is broadside on; or the length of the boat and the diameter of the turning circle may be determined in any convenient manner.

A special form of alidade has been devised by M. Risbec to be used on plane-tables for determining manœuvring properties. The pencil on the alidade is moved a short distance or step by an electromagnet controlled by a clock or chronometer, so that the observer has only to keep the sights trained on the object he is observing. Between the steps the pencil draws a circular arc, and the length of the arc between steps determines the change of direction, while the chronometer indicates the elapsed time.

Sailing-ships.—The control of sailing-ships depends mainly on the manner in which the sails are set and trimmed, and consequently almost all sailing-ships are steered by hand, even when they have large displacements. It takes more skill to steer a sailing-ship than a steamer, because it is necessary to keep the sails properly filled while the wind varies in force and direction.

The treatment of the problem of applying sails to a ship is entirely arbitrary, depending on experience. It is probable that sailing-ships reached their highest development in the era of fast-sailing clippers. Since that time ships have been built of steel instead of wood, and have had various devices to reduce the relative size of the crew required, but it is doubtful if there has been any real advance in the art of design. The recent advance in the power and speed

of sailing-yachts has been due in part to methods of construction and in part to a real development of the art of design.

Fore-and-aft and Square Sails.—There are two distinct types of sails used: those hung from yards square to the mast and called square sails, and those hung on stays or gaffs and booms, called fore-and-aft sails. All modern sailing-vessels carry some fore-and-aft sails; a full-rigged ship has jibs for her head-sail, and a spanker or fore-and-aft sail on the mizzen; she has also various stay-sails. Yachts and coasting-vessels commonly have fore-and-aft sail only.

In general, fore-and-aft sails are better for head winds, and square sails for fair winds. A square rig is almost essential for successful ocean navigation, where sailing routes are largely controlled by trade winds that blow steadily for months.

The large development of three- and four-masted schooners for coastwise trade is due partly to the prevalence of variable winds and partly to the fact that such a rig enables a small crew to handle a large vessel. Schooners are now built with six or seven masts.

Lifting- and Driving-sails.—Almost all sails are so set that the pressure of the wind gives a vertical component which, according to the nature of the sail, may be upward or downward. Square sails on a vertical mast have a slight lifting effect; on a mast with a rake such sails have somewhat greater vertical effect. Jibs and stay-sails have a lifting component; fore-and-aft sails set on a boom and gaff have a downward component. On the whole the total downward or upward component for the entire vessel is not important on most modern ships. The lateen sail, found in the Mediterranean and in Eastern waters, has a decided lifting effect.

A curious result comes from the discussion of the action of the wind on a lateen sail, in that it appears that under some circumstances the inclination of the boat does not increase with increased wind pressure. It appears that the total wind pressure may be resolved into three components: (1) a fore-and-aft component which drives the vessel; (2) a transverse component which tends to incline the vessel; (3) a vertical component which tends to lift the vessel. When the vessel has attained some inclination under the action of the second component, the resultant of the second and third components, i.e., the transverse and the lifting com-

ponents, will be a force lying in the sheer-plane of the ship, and if the mast has no rake it is a force acting along the mast tending to pull it out of the ship, but not tending to give additional inclination. It is said that such a rig allows the vessel to carry a large spread of sail without a great inclination, but that it requires delicate handling and is especially dangerous if the sail is taken aback by the wind getting on the wrong side of the sail. The lateen rig is heavy and requires a large crew compared with the usual fore-and-aft rig of coasting-vessels. A converse of this discussion will be found for a cat-boat which has but one fore-and-aft sail on a mast near the bow. Such a rig gives a strong downward component which may depress the bow dangerously when running before a strong wind.

Balance of Sail.—If a sailing-ship is to steer well, it is essential that the real centre of effort of the wind on the sail shall be vertically over the centre of resistance of the hull to lateral motion. Unfortunately neither centre is known, consequently we must depend on empirical methods.

In the discussion of the balance of sail it is assumed that all sail is trimmed flat in a fore-and-aft plane, that is, all sails are so drawn on the sheer-plan. The centre of effort of each sail is assumed to be at its centre of figure. The centre of pressure of the whole sail-plan is assumed to be at the centre of gravity (or centre of figure) of the whole system of areas. This point is most readily found by taking moments about convenient horizontal and vertical axes. Thus the area of each sail is multiplied by the distance of its centre of figure from the assumed axis, to find its moment with regard to that axis and the sum of the moments of the several sails is taken to find the total moment about that axis; this total moment is divided by the total sail area to find the distance of the centre of pressure from the axis.

The centre of pressure, especially if there are few sails, is often found by the following method: The line connecting the centres of figure of two sails is divided inversely proportional to their areas, thus finding the centre of gravity (or centre of figure) of the two sails combined; this centre is now connected with the centre of figure of a third sail, and that line is divided inversely as the area of the one sail to that of the two sails already combined, thus

finding the centre of figure for the three sails. The process can now be extended to a fourth sail, etc. It evidently is very tedious and uncertain when applied to a full-rigged ship.

In balancing sail only plain sail is considered. Thus for a full-rigged ship there will be included the courses, the topsails and the topgallantsails, the jibs, staysails, and the spanker. The royals and studding-sails are not included in this calculation; but now that double topsails are in common use, studding-sails are seldom used.

The American type of three- and four-masted schooners carry only plain sail, which is all included in the above calculations. That is, there will be included the fore-and-aft sails, the gaff-topsails and the jibs, and also the staysails.

The rig of a yacht for either racing or cruising should be carefully balanced for several contingencies, such as with all sail set except the spinnakers, with plain sail only, and with the sail reduced or reefed. The final adjustment of the sail must be made by trial.

The centre of lateral resistance is probably somewhat ahead of the centre of figure of the immersed area of the sheer-plan, especially for full ships that make a good deal of leeway under sail. On the other hand it is pretty certain that the real centre of effort of the sail is ahead of the centre of area as found above. Consequently a correct balance of sail can only be obtained by trial or by comparison with other ships. It is customary to set the centre of area of the sail some distance ahead of the assumed centre of lateral resistance. For old full men-of-war and merchant-ships the centre of area of the sail was frequently one-quarter of the length of the ship ahead of the centre of lateral resistance. For more modern ships it is one-twentieth or less. Small fine vessels may have the centre of effort directly over the centre of lateral resistance, or the centre of the sail may even be aft of the centre of resistance.

The final balance of sail for any craft must be made by the way in which the sails are trimmed; it should give a small tendency to come up into the wind, which tendency is resisted by the rudder. This tendency, called ardency, is most affected by the head-sail and the aftermost sail because they are remote from an axis through the centre of lateral resistance. The contrary tendency, called slack-

ness, is likely to be dangerous, and is always to be avoided. Ardency normally increases as the strength of the wind increases, because the centre of application of the resultant wind pressure is carried farther to the leeward as the ship heels farther over; consequently the ship tends to come up to the wind as the strength of the wind increases, and thus tends to spill the wind out of the sails and relieve the ship; if a ship is slack, it tends to fall off as the wind freshens, and this may give a dangerous inclination.

Power to Carry Sail.—The power of a ship to carry sail should properly be treated in connection with the discussion of propulsion, but it is convenient to give it here together with the method of determining the centre of pressure of the wind on the sails.

The amount of sail that may be properly given to any sailing-craft is determined by experience, depending on the service for which it is intended. Thus racing-yachts carry much more sail than cruising-yachts do, and they have more sail than would be safe for fishermen. Sailing-ships of large size, however, have the amount of sail limited by the height of masts that can be given to them, and could carry more sail if any way could be devised for spreading it.

The amount of sail that is proper for a ship may be assigned in the following arbitrary manner: Assume all the sails to be trimmed flat fore-and-aft as in the determination of the centre of lateral resistance, and find the vertical location of the centre of pressure by a method similar to that already described for finding the longitudinal position of that point; in practice the two determinations are made at the same time. If the pressure of the wind is p pounds per square foot, and A is the sail area in square feet, while h is the distance in feet of the centre of pressure above the centre of lateral resistance, then the inclining moment of the wind may be taken to be

$$pAh;$$

it appears as though this moment should include a trigonometrical function of the angle of inclination, but for the angle at which a ship will sail in a fair breeze that function is nearly equal to unity. This inclining moment may be equated to the righting moment of the ship in foot-pounds, that is, to

$$2240D(r_0 - a)\theta,$$

where D is the displacement in tons, $r_0 - a$ is the metacentric heigh in feet, and θ is the angle of inclination at which the ship is expecte to sail. The area of sail consequently becomes

$$A = \frac{2240 D(r_0 - a)\theta}{ph} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

If θ is made 20 degrees, the value of p as determined fron practice appears to be for

Small racing yachts. 0.9
Large cruising yachts. 1.5
Merchant ships. . . . . 2.0 to 2.5

The largest value for merchant ships is found for those ships tha have the sail area limited by the height of the masts.

The following table gives data for a few ships.

SAILING-SHIPS.

	Length, Feet.	Beam, Feet.	Depth, Feet.	Draught, Feet.	Dis-place-ment, Tons.	Plain Sail, Sq. Ft.	Total Sail, Sq. Ft.
Three-masted ships:							
Thermopylæ.	210	36	. . . .	. . . .	1970	17500	21200
	267	39.3	23.3	20.2	3407	23500	38500
	197.5	32.2	19.9	17.3	1775	10980	17300
Schooners:							
Douglass, 6 masts. . . .	306	48	29.8	24	7700	. . . .	36000
J. B. Prescott, 5 masts	290	43.7	25.5	23	6400	21500	27000
Marion F. Sprague. . .	151	35					
Fishing schooner.							
Rob Roy.	86	23.1	. . . .	14.3	173	5220	7970

CHAPTER XIV.

WEIGHT AND STRENGTH.

IN order to ensure that a ship shall have sufficient buoyancy and stability, and that it shall have the proper trim, it is necessary to make calculations of the weight of the hull and its contents and of the position of the centre of gravity.

The principles for these calculations are very simple; the calculations are likely to be long, laborious, and uncertain. From the plans and specifications the size and weight of every important member of the ship can be determined; it is sufficient, then, to make detailed calculations for the weight of each member and sum up the total to get the weight of the hu l, and similar calculations can be made for the machinery, fittings, and contents of the ship. For the determination of the centre of gravity moments may be taken of all members about a convenient axis; the total moment divided by the total weight gives the distance of the centre of gravity from that axis.

A considerable portion of the weight of a ship must be estimated by direct comparison with other ships; for example, the paint may weigh many tons on a large ship; fortunately such secondary weights are so widely distributed that they have little influence on the centre of gravity of the ship.

All important vessels should be weighed during the process of construction; that is, all material worked into the ship is weighed as it is added, and all scrap and refuse from the ship are weighed as they are removed. In order that the information thus obtained shall be valuable, the weighing must be done systematically under the direction of some competent and responsible person.

Finally the calculations are systematized and average weights are found and tabulated from which rapid estimates of probable

weights of new ships can be made. Thus the percentage of displacement that must be allowed for the hull is well known for all classes of ships, as well as the percentages for the weight of engines, boilers, coal, and cargo.

Since the hull of even a large ship is framed after a comparatively simple system, it is possible to so systematize the calculations for weight and centre of gravity of the hull that the computations shall not be excessively laborious.

Calculations by Sections.—It is convenient to assimilate the calculations for weight and centre of gravity, to the determinations of displacement and centre of buoyancy, of the carene. Thus the displacement of the U. S. Lightship is determined by aid of Fig. 27, page 33, in which the base-line is the length of the ship and the ordinates of the curve of areas are the areas of transverse sections; the volume of the carene is consequently represented by the area of that curve. It is clear that an ordinate of the curve of areas may be made to represent the volume of a transverse slice of the carene, which is one foot long, and that a similar curve can be constructed by using for ordinates the weights of such transverse slices. For the present purpose it is customary to replace the curves of transverse areas by a curve of buoyancy, which has for its ordinates the buoyancy of such transverse slices. These ordinates can be obtained by dividing the entire areas of transverse sections of the carene by 35, the number of cubic feet of salt water per ton. Fig. 214, page 584, gives such a curve of buoyancy for a ship in quiet water and another curve of buoyancy among waves, which latter curve need not be considered now. In calculations of strength it is customary to use the curve of buoyancy, but for the determination of the weight and the vertical position of the centre of gravity of a ship it is customary to use tabular methods that have a general resemblance to displacement sheets, though in detail and appearance they are very different. The object of the present discussion is to give an idea of the general methods of calculations of weight and centre of gravity, for which purpose a comparison with the graphical methods for finding displacement and centre of buoyancy will be found instructive. A graphical determination of weight may be made by aid of a diagram having the length of the ship for its base, and for its ordinates the

weight of the ship per foot of length in tons. Fig. 214 has such a curve of weights with an irregular contour due to the concentration of certain weights like engines, boilers, guns, and turrets, masts, etc. The total area of the curves of weights must be equal to the area of the curve of buoyancy. If the ship has overhanging ends the curve of weights will extend beyond the perpendiculars at the bow and stern.

Concentrated weights may be determined by calculation from drawings, by comparison with ships already built, or by actually weighing the parts. Weights of such parts as engines and boilers are usually assumed to be uniformly distributed over their lengths. Weights of masts and some other concentrated weights may be arbitrarily assumed to be distributed over one or more frame spaces. The weight of the general structure of the hull is assumed to be distributed over the whole length of the ship, and the weight per foot of length is determined at intervals. Since the general construction of the hull is more or less continuous it is possible to so systematize the calculation of its weight as to reduce the labor. The usual method, called the method of sections, is to take a transverse section of the framing at each tenth or twelfth frame. All the longitudinal members cut by this section, including the external shell plating (with butt-straps and liners) and inner bottom and decks, are noted, and their weights per foot of length are obtained from the specifications of the ship or from tables of scantling. The longitudinal members will include the keel, keelson, bilge keelsons, stringers in hold, and the shelf or stringer under the beams. Such members are either continuous, intercostal, or partly continuous and partly intercostal. Continuous members like angles, channels, I beams, and bulb beams are readily treated as their weights per foot of length are known. Near the middle of the ship this weight can be used directly, but near the ends the length of the number per foot of length of the ship is greater than unity. Its true length is to be obtained by multiplying by the secant of the angle it makes with the axis of the ship; this secant may be as large as 1.1, and cannot be neglected without serious error. Fortunately longitudinal members are rarely horizontal, and the angle can be measured on the general framing plans. The plating, clips, and other

with the weight of the frame, and finally re
per frame-space to the corresponding weight

Frames.—The transverse frame of a small angle-bar, reverse frame, and a floor-plate. T frame from its upper end to the keel may be me of the ship, and may be multiplied by its weigl the reverse frame may be treated in the sa doublings can be calculated separately. The a measured, allowing for lightening-holes, and mu per square foot. The deck-beam may be measu side, allowing for the part turned down for the there is a plate bracket it can be allowed for length and weight per foot the weight of the If there is a beam for every other frame, the the beam is allowed for each frame-space. Th frame divided by the length of the frame-spac weight per foot of length.

The simplest form of framing is taken in sake of brevity, but the same method may be ap larger and more complicated framing, having mo will of course involve more labor. Large w widely spaced must have their weights divided frame-spaces between them and by the length feet.

Plating.—As an alternative method, the she

but as the moment must be determined also, the present method must be used at least for that purpose.

Decks.—Weights of decks can be readily computed from the deck-plating plans where sizes and scantlings are correctly represented. Proper allowance should be made for liners, butt-straps, and rivet-heads.

Wooden decks are laid over tie-plates and the deck stringer-plate at the side. The stringer-plate and tie-plates and other steel construction are to be calculated separately. Then the wooden deck is to be calculated from its thickness and weight per cubic foot, which weight should include weight of bolts and other fastenings.

Butts, Liners, and Rivets.—Although it is possible to calculate and allow for butt-straps, or laps at ends of plates, for liners under the outer strakes, and for rivets, it is convenient and customary to allow for all these items by arbitrary percentages, determined from practice.

Cement.—All parts of the framing and plating against which bilge-water is likely to lie should be protected by hydraulic cement. The cement on the bottom plating may be laid on as thick as the depth of the angle-bar above which limber-holes for drainage are cut. If the floor is flat, this thickness may be carried to the turn of the bilge. Where there is a rise of the floor, the thickness of the cement is diminished toward the bilge and may finally be a wash laid on with a brush. There are two ways of determining the weight of the cement; an attempt may be made to determine the volume from the thickness and area of the cement, which volume is then multiplied by the density; allowance must be made for cement wash and other cement which is not readily calculated; sometimes the total weight of cement is estimated in a lump and the distribution is made more or less arbitrarily. Modern practice is to replace the heavy cement by a comparatively thin coating of water-proof material.

Diagram of Weights.—Having the weight of the hull per foot of length for a sufficient number of stations, the diagram of weights like that in Fig. 214 can be constructed, using for the base-line the length of the ship, and for the ordinates the weights per foot of length. It is convenient to draw first a curve to represent the weight of the hull, using as ordinates the weights per foot of length calculated by the

method just described, after which a smooth curve may be drawn for the weight of the hull. As already noted, this curve is likely to extend beyond the ends of the base-line, more especially at the stern.

The curve of the weight of the hull does not include the weights of the stem and stern-post, nor of bulkheads, deckhouses, masts, spars, etc. These are usually calculated by the usual rules of mensuration or estimated from known weights for other ships, and are treated as concentrated loads along with the machinery. It is, however, customary to make a special determination of the vertical and longitudinal location of the centre of gravity of the hull, not including these concentrated weights, and another calculation taking account of them, for use in design of future ships, as well as for the adjustment of weights to give proper trim. It is at once evident that the fore-and-aft location of the centre of gravity of the hull will be at the centre of figure of the curve of hull weights, and can be located in the usual way for a plane area. The discussion of the determination of the vertical location of the centre of gravity of the hull will be reserved.

The hoisting-engine, bitts, windlasses, anchors, etc., are usually associated with the hull. Their weights can be obtained from catalogues of makers, or may be determined from comparison with ether ships. The propeller is properly a part of the propelling machinery, but it is sometimes associated with the hull, and its weight is in such case estimated by comparison with that of another ship.

Propelling Machinery, Cargo, etc.—The construction of the curve of weights is to be completed by adding the weights of the propelling machinery, bunker coal, stores, cargo, and trimming-tanks, and for war-ships the weight of the armor and armament.

The propelling machinery includes the engines, boilers, propellers, and shafting. together with auxiliary machinery and appurtenances. The locations of these several parts can be determined from the machinery drawings and their weights, and centres of gravity may be taken from the determinations of the engine draughtsmen, or they may be estimated by comparison with known ships and their propelling machinery. The bunker coal can be allowed for from the size, form, and location of the bunkers, and stores can be treated in a

similar way; trimming-tanks should be considered at the same time, but may require modification later. Finally the cargo can be estimated from the dimensions of the holds and the probable density of the cargo. It is customary as a rule to assume that the holds are entirely full of homogeneous cargo; this part of the weight calculation for a ship in general service is most unsatisfactory, as the amount, density, and stowage of the cargo vary with the service and can seldom be foretold.

Now the total weight of the ship and her contents must be equal to its displacement, consequently the sum of all the calculated weights ought to be equal to the displacement. True, there are a good many weights, such as fittings, paint, etc., which have not been considered, and which amount to many tons in a large ship. Such weights must be estimated from experience and comparison with other ships. They are assumed to be uniformly distributed, since any attempt to allow for distribution is unsatisfactory, if not impossible.

After all known and estimated weights are allowed for, the difference between the weight and displacement is to be added to or taken from the cargo before proceeding with the further calculations. If the discrepancy is large, it shows that the design of the ship is inadequate and may call for a new design.

The diagram of weights is to be completed by adding areas showing the location and amounts of the several weights representing the burden of the ship, such as propelling machinery (engines, boilers, etc.), coal, stores, and cargo. The resulting diagram is bounded by an irregular line with protruding areas where weights are concentrated and with gaps where there are spaces required for cabin accommodations below decks or for working the ship. The area of the figures bounded by the irregular line represents the weight of the ship. It can be readily obtained by aid of an integrator, or it may be calculated by any suitable method of mensuration. Some designers draw a smooth curve having the same area between it and the base-line and calculate the area by the trapezoidal rule.

When careful calculations are made it is better to make direct numerical determinations of the several parts of the weights as described and to take the sum for the total weight, and to use the determination from the area of the diagram of weights as a check only.

The diagram will be further considered in the calculations for strength.

Trim Calculation.—In the discussion of the weight of the hull its fore-and-aft centre of gravity was assumed to be determined; when other weights, such as propelling machinery, stores, and cargo, are taken into consideration their longitudinal location must be determined; finally by taking moments about a convenient axis, the centre of gravity of the total weight can be determined. This centre of gravity should be over the centre of buoyancy as determined from the displacement calculation. If the two centres are not on the same vertical line the ship will be out of trim. The effect of the discrepancy is determined by aid of the moment to change trim an inch. If the discrepancy is small it may be neglected, being assumed to be provided for by stowage of cargo or by changing trimming-tanks, or it may simply be ignored. If the discrepancy is large, giving an undesirable change of trim, then some of the weights in the ship must be shifted until a satisfactory trim is obtained. Trim calculations are of the greatest importance and every precaution must be taken to avoid blunders which can lead to gross errors. Not unfrequently weights are shifted after a design is complete, or nearly complete, and then the designer must make sure of such compensation as will secure the proper trim.

Vertical Position of the Centre of Gravity.—At the same time the calculations are made of the weight of the hull and its contents determinations should be made of the vertical positions of the centres of gravity of the several parts. This may be conveniently separated into the determination of the centre of gravity of the hull and the determination of the centre of gravity of the propelling machinery, stores, and cargo. For the determination of the centre of gravity of the hull we may follow the general method for the determination of the weight of the hull per foot of length. For this purpose we will take moments of each member (angle, channel-bar, or plate) about the base-line at the keel, the arm being measured on the proper drawing; if the ship has an external keel the axis may be taken at the top of this keel; if there is a flat plate keel the axis will be at its bottom. Longitudinal members, such as keel, keelsons, stringers, etc., are easily disposed of. Trans-

verse members may require more consideration; for example the curved angle-bar forming the frame of a simple ship extends from the keel to the upper deck. To get its centre of gravity it may be divided into a number of equal parts each of which is nearly straight, with the centre of gravity at its middle. Join the centres of gravity of adjaçent parts by straight lines and bisect the lines for the centre of gravity of the two parts thus linked together. Now join these secondary centres of gravity and get a set of tertiary centres of gravity, and so on till a single centre of gravity is found. If the parts cannot be joined in equal pairs, the line joining the centres of gravity of parts to be combined must, of course, be divided inversely as the weights of those parts. Other transverse members and shell plating can be treated in the same way. A diagram of moments similar to the diagram of weights of the hull can now be plotted with the same base-line, using for ordinates the moment of a foot of length of the hull; or the computation can be arranged in a convenient tabular form, from which the total moment can be determined.

After the vertical position of the centre of gravity of the longitudinal and transverse framing and shell plating and decks has been found, the concentrated weights like stem and stern-post, rudder, bulkheads, etc., are to be taken into account to determine the centre of gravity of the structural hull. After this, superstructures, masts and spars, hoisting-engines, and anchors, may be considered, and a new centre of gravity may be found. Next comes the consideration of fixed loads, including engines, boilers, shaft and propeller, etc. Commonly the boilers are treated both as full and as empty. Finally we add to our calculations coal and stores, after that the cargo, assuming various density and stowages if that is considered desirable. A complete calculation for vertical position of centre of gravity should enable us to determine the position of the centre of gravity and the metacentric height for all conditions of the ship, such as (1) when light and without water in the double bottom or other water-ballast tanks, (2) when light but with water-ballast tanks filled completely or in part, (3) when provided with coal and stores but without cargo, (4) when loaded in one or more ways. This number of conditions may be increased if necessary, or some of those mentioned may be omitted.

Similar calculations for war-ships are made for the ship when light (as for docking), when supplied with normal amount of ammunition and stores and coal, and when supplied with all the coal the bunkers will hold, and sometimes for other conditions.

It is well, though not always necessary, to calculate the trim of the ship for all the conditions named.

Calculation for Strength.—After the weight and trim calculations are complete we are ready to make calculations for strength, following certain arbitrary methods. It is customary to make calculations for the ship in quiet water and when placed on a wave of the same length. The stresses in quiet water when the ship is properly designed and loaded are likely to be small, except for ships which have the machinery way aft and a large trimming-tank forward. The calculation for a ship on a wave is purely conventional, since it is not certain that the ship will ever be placed on such a wave in service, nor that the greatest stress in service will be equal to that found from the calculation. Nevertheless the conventional calculation will give relative results from which we may infer with some degree of certainty whether or not a ship is likely to be strong enough to endure the service for which it is designed.

Though ships have undoubtedly been lost at sea because they were weak, it is seldom that a carefully designed ship fails entirely even though her design is new and to a certain extent experimental. Weakness is usually shown by working of rivets in the shell plating, and consequently all ships should be carefully inspected when in dock for such evidences of weakness. Sometimes weak ships are distorted and are seen to be unfair when docked. The weakness may be structural or local, and may call for addition to the general structure or for local strengthening or stiffening.

The general methods of calculation are similar to those for a continuous girder, except that no attempt is made to determine deflection. The process is to determine the loads or weights as by the preceding discussion and the supporting force of buoyancy; from these the shearing forces at all sections of the ship may be determined, and finally the bending moments at desired sections can be found, and therefrom the stresses in the most strained members.

Shearing Forces.—Fig. 214 represents by the irregular line the curve of weights of a first-call battle-ship, and on it is drawn also the curve of buoyancy of the ship in still water. The difference between the ordinates to these curves at any point is the effective load per foot of length at that point. If the ordinate for the curve of weights is the larger, the difference is considered to be positive. Taking ordinates at a sufficient number of points, we may draw a continuous curve of loads; to avoid confusion this curve is omitted from Fig. 214. This curve is partly above and partly below the base-line; the first case will occur for those parts of the ship where the weight exceeds the buoyancy, and the second where the buoyancy is the greater. The curve crosses the base-line at points where the weight per unit of length is equal to the buoyancy per foot of length. If we consider that the area between the curve of loads and the base-line is positive when the curve is above the base-line, and negative when below, then the algebraic sum of these areas is zero; this gives a valuable check on the trim calculations, but cannot be substituted for them when the trim of the ship is important. The areas are most easily measured by aid of an integrator running on a track parallel to the base-line. If the integrator is set with the area wheel at zero at the beginning of the curve of loads, it should be zero again at the end of that curve.

To proceed with the determination of the shearing forces acting on the hull of a ship, it is convenient to assimilate half of the ship to a cantilever, for which the shearing force at any section may be determined by summing up all the loads between that section and the end of the cantilever. Concentrated loads or uniformly distributed loads on such a cantilever may be determined separately and added to find the shearing force, but an irregularly distributed load like that on a ship, as shown by the curve of loads in Fig. 214, must be obtained by graphical integration. For this purpose it will be most convenient to draw the integral curve by aid of the integraph described on page 26; the tracing-point may follow the irregular curve of loads, and the recording-point will draw the curve of shearing forces, for which the scale of ordinates will be determined by the way the instrument is set. The instrument is most conveniently used by starting at the left and tracing toward the right;

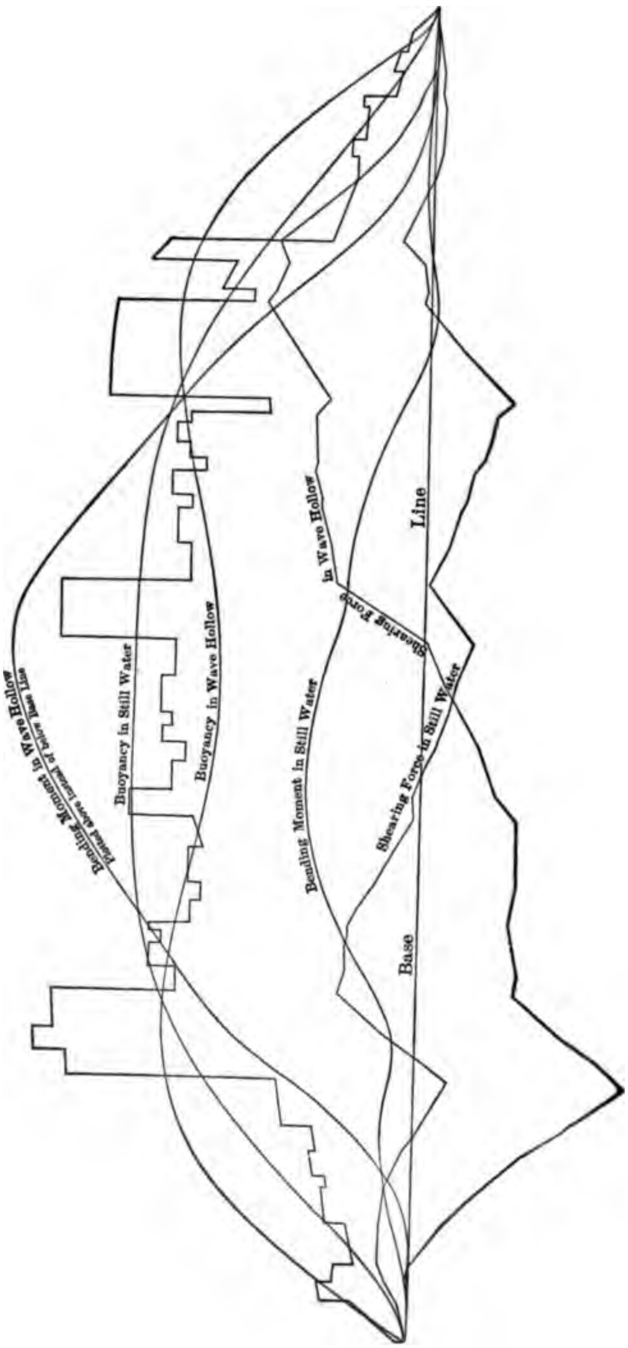
Bending Moment in Wave Hollow
Plotted above instead of below Base Line
Buoyancy in Still Water
Buoyancy in Wave Hollow
Bending Moment in Still Water
Shearing Force in Wave Hollow
Shearing Force in Still Water
Base Line

FIG. 214.

the curve of shearing forces in Fig. 214, which was constructed in another way, is drawn from the right, but the inversion once noted need not lead to any confusion.

If there is no integraph at hand, an integrator or a planimeter may be used to measure the area of the curve of loads, and the area, beginning at the end of the ship and extending to a certain ordinate, is proportional to the shearing force at that ordinate. If an integrator or planimeter cannot be had, the areas may be computed by any convenient method.

The comparison of the half-ship to a cantilever is convenient because there is no supporting force at the end of the ship. If desired, the whole ship may be compared to a beam supported at the middle, and the integration for the shearing forces may be carried from each end to the centre of gravity of the ship; or, more conveniently, the integration may proceed from end to end of the ship.

The curve of shearing forces will cross the axis or base-line at one or more points, at which points there is, of course, no shearing force on the hull.

Bending Moments.—It is a well-known principle of the theory of beams * that the bending moment at any section is represented by the expression

$$M = \int s dL,$$

where M is the bending moment, s is the shearing force, and L is the length of the beam.

To find the bending moment at any section of a ship it is sufficient to integrate the curve of shearing force from an end of the ship to that section, that is, to measure the area between the shearing force and the base-line up to the point at which the section is taken; or the integral curve can be drawn directly by aid of an integraph.

The bending moment reaches a maximum at the ordinate for which the curve of shearing forces crosses the base-line. In Fig. 214 the curve of shearing forces in quiet water crosses the axis near the middle, and the maximum bending moment is found at the same

* Lanza, Applied Mechanics, pages 317 and 745.

place. If the shearing-force curve crosses the axis more than once, there well be more than one maximum, of which the largest, regardless of the sign, will be taken for the computation of stress in the framing of the ship. A tendency to bend a ship upward at the middle is called a hogging strain, and a tendency to bend it downward at the middle is known as a sagging strain. A curve of bending moments above the base-line indicates the existence of a hogging strain, and a curve below the axis shows a sagging strain; if the curve crosses the axis, the parts above and below the axis correspond respectively to hogging and sagging strains. The ship for which the curves of Fig. 214 are drawn is subjected to a sagging strain when in the hollow of a wave, but for convenience the curve is transferred to the upper side of the axis. A loaded merchant-ship has usually a hogging strain from end to end in quiet water; unloaded it may have a sagging strain at the middle, due to the concentrated weight of the engine and boilers. Some of the old turreted war-ships had complicated curves of bending moments in quiet water, with several changes from hogging to bending on account of the concentration of weights at two turrets near the ends and at the machinery space amidships. A modern armored vessel has the armor and armament more distributed, and will be likely to have a hogging moment in quiet water.

Ship on a Wave.—The conventional calculation for strength of a ship among waves is always made for a wave having the length of the ship on the water-line and a height equal to one-twentieth of its length. No attention is given to the dynamic action of water affected by a wave; for this effect, though appreciable, is not large and affects all ships in much the same way; consequently the added complication is not profitable, more especially as calculations for strength on a wave do not give absolute results, and for purposes of comparison the simpler method is better.

The essential feature of the calculation for a ship on a wave is the substitution of a conventional trochoid for the usual horizontal water-line. It would appear that equally good results would be obtained if the contour of the conventional wave were assumed to be a curve of sines. Each section of the carene is terminated by a horizontal line at the height where the section is

cut by the trochoid, as indicated by Fig. 215. Calculations for buoyancy and centre of buoyancy and trim are made as for quiet water except for the difference just pointed out. The weight of the ship and its distribution are not influenced by placing the ship on a wave, and the curve of weights is consequently unchanged.

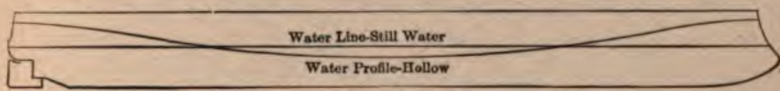

FIG. 215.

The first operation is to draw a trochoid on the sheer-plan which shall give the ship the same displacement and trim. This can be done by trial only, and for this purpose Bonjean's curves will be found convenient. It is probably not profitable to be too particular, since the trim has not the importance it has in quiet water, and extreme accuracy is out of place in a method which gives relative results only.

After the ship is placed on the wave a new curve of buoyancy must be drawn and afterwards curves of loads, shearing forces and bending moments. The ship is usually calculated with the crest of the wave at or near the middle of the length; sometimes a calculation is also made for a hollow at the middle of the length. Merchant-ships loaded always show small strains when the hollow of the wave is at the middle of the length, because the deeper immersion of the bow and stern counteracts the weights of the ends. A merchant-ship unloaded or a war-ship with turrets may have large strains with the hollow of the wave at the middle. Fig. 214 shows the curve of bending moments for a ship with a wave hollow at the middle; there is a sagging strain from end to end; this curve is drawn above the base-line as a matter of convenience, though its ordinates are properly negative. The curve of bending moments with a crest near the middle of the ship has the same general character as the curve shown for quiet water, but with a much larger maximum moment; it is omitted from the figure to avoid confusion of the diagram.

Calculations are almost always made for the ship erect, but the same process may be applied for the ship when heeled to various

angles, or for a ship laid diagonally across waves. But these latter calculations are much more intricate and laborious and therefore they have been made only for a few cases.

Wave Pressure.—In the conventional method of calculations for a ship on a wave the pressure of the water is tacitly assumed to be the ordinary hydrostatic pressure due to the depth of the water. But in the theory of waves it appears on page 263 that the pressure for any wave surface is constant and can be calculated in the form of the equivalent head of water by the formula

$$H = D - \frac{\pi r_0^2}{L}\left[1 - e^{\frac{4\pi D}{L}}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where L is the length of the wave from crest to crest, r_0 is the radius of the orbit of a particle at the surface, and D is the vertical distance of the centre of the orbit of a given particle below the centre of the orbit of a particle at the free water surface; e is the base of the Naperian system of logarithms.

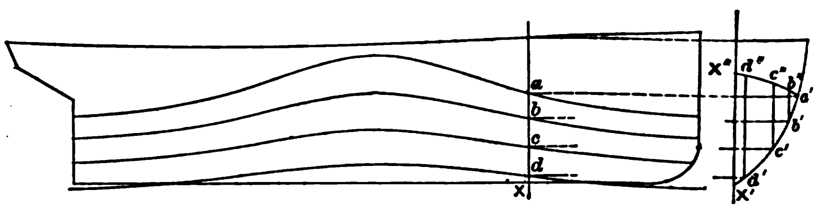

FIG. 216.

Fig. 216 represents a ship on the crest of a wave, and a number of wave surfaces at varying depths; to bring the feature under discussion into prominence the ship and wave are made disproportionately short and high. The radius for the orbit of particles belonging to each of the wave surfaces is computed by the equation

$$r = r_0 e^{\frac{-2\pi D}{L}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the pressures for the wave surfaces are computed in heads of water by equation (1).

Having drawn a sufficient number of wave profiles and calculated the wave pressures by equation (1), we may proceed as follows to find the effective displacement. For a given section, as at x on the sheer-plan, note the points of intersection by the wave surface at a, b, c, d, etc., and also at a', b', c', d' on the half transverse section. At these points erect verticals $b'b''$, $c'c''$, $d'd''$, etc., each equal to the equivalent static pressure H_b, H_c, H_d, etc., calculated by aid of equation (1). Draw a curve $a'b''d''x''$; then the upward pressure of the water for one foot of length is equal to the weight of a volume of water which has the area $x'a'x''$ and a length of one foot. That is, we may use the area $x'a'x''$ in place of the usual area for a half-section in calculating the real buoyancy for the ship on a wave. The curves of areas and the displacement and trim are to be determined in the usual way by aid of an integrator.

In Fig. 216 the section $x'a'x''$ is in the hollow of the wave and the curve $a'b''x''$ is above the horizontal line $a'a$; on a crest the curve corresponding to $a'x'x''$ is below the horizontal line. Consequently the buoyancy of a ship on the crest is less than that determined by the conventional method, and the buoyancy at a hollow is greater; that is, the conventional method exaggerates the effect of placing a ship on a wave, the error being from 5.5 to 7.5 per cent.

Moment of Inertia.—Having found the section of a ship at which the greatest stress due to bending is likely to be found, whether in quiet water on on a wave, we must find the moment of inertia of that section of the framing of the ship by substantially the same method as is used for a rolled beam or a built-up girder.

To find the moment of inertia select a convenient axis either at the top of the external keel or the bottom of the keel-plate or at the middle of the height of the section. Take account of all longitudinal members cut by the section, including keel, keelsons, stringers, shell plating and strength decks. Multiply the area of the section of each member by the distance of its centre of gravity from the assumed axis and sum the results for the moment of the section about that axis. Again, multiply the area of each section by the square of the distance of its centre of gravity from the assumed axis, and sum the results for the

moment of inertia of the section about that axis. Find the centre of gravity of the section and reduce the moment of inertia to an axis through this centre of gravity. Should any member of the section have an appreciable moment of inertia about its own centre of gravity, that quantity should be added to the sum for the moment of inertia of the section; this is necessary as a rule for vertical members only. The areas of plates can be determined from their width and thickness or weight per square foot. The areas of rolled forms, such as angles, channels, bulb beams, etc., can be found in makers' catalogues or may be inferred from their weight per unit of length.

In making calculations for the moment of inertia of the section of a ship's frame the area cut away by all rivets-holes in that section should be deducted. Computers commonly deduct rivet-holes from members in tension, but not from members in compression, under the mistaken notion that the rivets fill their holes completely and transmit stresses as though there were no holes. This clearly cannot be correct for the rivets are driven hot in cold plates and contract after cooling. Moreover, tests on riveted joints in compression, though few and insufficient, indicate that all rivet-holes should be deducted both for tension and compression. This method is, moreover, the simpler, especially when the same section is calculated for hogging and sagging.

Calculation of Stress.—After the bending moment at a given section and the moment of inertia for that section have been determined, the stress can be found by the usual equation,

$$\sigma=\frac{My}{I},$$

where M is the bending moment in inch-pounds, I is the moment of inertia in inches, and y is the distance of the most strained fibre from the axis, also in inches. The bending moment M will commonly be expressed in foot-tons on the diagram of Fig. 214, and must be properly reduced; I and y should be determined in the required units directly.

The following table of allowable stresses was prepared by Mr. W. John*:

MAXIMUM STRESSES ON UPPER WORKS OF SHIPS.

Tonnage.	Stress, Pounds per Square Inch.	Tonnage.	Stress, Pounds per Square Inch.
100	3700	800	10300
200	5300	900	10700
300	6900	1000	11600
400	8000	1500	12000
500	8800	2000	13200
600	9600	2500	15800
700	10200	3000	18100

If any calculation shows excessive stress for any ship, the ship should be strengthened by adding members near that member which has the greatest stress. This will usually be near the highest deck, which is included in the calculation of strength; for the framing of floors and double bottoms (to give requisite strength and rigidity to carry machinery and cargo, and to allow of docking) will give more than strength enough at the bottom of the ship. Often a ship may be improved by transferring material from the bottom to the strength deck. The strength deck should be heavy enough to serve for the upper chord of the girder, and should be well fastened and made truly continuous. Where the deck is cut for engine-trunk and cargo-hatches compensation must be allowed in form of plate stringers or otherwise.

It is important that all light shade-decks, shelter-decks, and deckhouses shall be discontinuous, so that they may not enter into the longitudinal structure of the ship. If a transverse interruption of a shade-deck would cause inconvenience by admitting water, a slip-joint may be devised which shall make an effective discontinuity of structure. In like manner bulwarks or other parts may be made to exclude the sea and add to the buoyancy without complicating the design for strength.

* Trans. Inst. of Naval Arch., vol. 15.

Strength of Bulkheads.—In addition to the investigation of the structural strength of a ship there are numerous calculations for local strength or for strength of certain members; but such calculations are usually special applications of the methods of applied mechanics, which are not essentially different from problems that are met in general engineering. An exception may, however, be made in the case of bulkheads, on account of the great importance of proper investigations of their strength, and on account of the fact that such investigations are commonly omitted in merchant work. The problems which arise require the application of the methods for continuous girders, with the peculiarity that the load is a uniformly increasing load due to the depth of the water which may act on the bulkhead when it is called upon to resist water pressure.

In discussing the strength of bulkheads, two distinct problems arise, namely, the strength of the framing which supports the plating, and the strength of the plating itself, for the rigidity of the framing of large and important bulkheads is so much greater than that of the plating that the latter can contribute little or nothing to the general or structural strength of the bulkhead. Indeed the plating is commonly so thin that it must be treated as though it were devoid of rigidity, just as a fibrous material would be if made waterproof and subjected to pressure. Properly the framing of a bulkhead should be independently secured to the framing of the ship; when the framing of the bulkhead consists only of angle-irons riveted to the plating and not extending to the edge, the security of the bulkhead will depend on the riveting of its edge to the framing of the ship, and there is likely to be a dangerous concentration of strains at that place, more especially as the calculations of stress and strain under such conditions are difficult and uncertain.

The framing of a bulkhead should consist of Z bars or other rolled or built-up forms which are adequately secured at their ends to the framing of the ship. The simplest frames, or stiffeners, will be made up of vertical bars, or other members, extending from the bottom of the hold to a deck, or from one deck to another. Such

vertical members will naturally be placed over the corresponding members of the framing of the ship. For example, the frames for a longitudinal midship bulkhead will each be placed over a floor, though there need not necessarily be a bulkhead frame over every floor-frame. At the present time all important steamships have double bottoms with a sufficient number of longitudinal members over which the frames for a transverse bulkhead can be placed and to which the lower ends can be adequately secured. There is likely to be some difficult in securing the upper ends of bulkhead frames, as the decks to which they are attached may lack rigidity.

Bulkheads near the ends of large ships beyond the double bottom, or in small ships which have no double bottom, may be difficult to deal with, since there may be no adequate means for securing the ends of the vertical bulkhead stiffeners. It may be sufficient to stiffen such bulkheads with angle-irons, or Z bars riveted to the plating, especially as the depth of the hold in small ships is correspondingly small, and since there is likely to be an additional deck or flat worked in near the bow and stern of a large ship.

If the framing of a ship fails to give adequate security at the ends of the frames of an important bulkhead, then the framing of the ship should be changed or strengthened to provide for the proper fastening of those frames. If this is not done, it is likely that the first time a compartment at one side of the bulkhead is filled, the bulkhead frames will distort the members to which they are fastened and start serious leaks, if indeed they do not tear away from their fastenings and cause a complete failure of the bulkhead.

In general it will be advisable to use widely spaced, deep frames for a bulkhead whenever practicable, since the thickness of the plating is likely to be controlled by the liability to corrosion quite as much as by the requirements for strength. Again, it is well, when convenient, to divide the length of a bulkhead frame in the hold by carrying a side stringer across a transverse bulkhead, or by providing a similar stringer on a longitudinal bulkhead, as the compound structure can probably be made the lighter.

It is convenient to discuss, first, the strength of a vertical bulkhead frame, and to consider it to be a beam having a uniform section fixed at the ends and subjected to a load which increases uniformly from the top downward, as represented by Fig. 217. If the depth of water producing pressure at the top is h_0, if the length of the frame is l, and if the space between successive frames is S, all in feet, then, representing the weight of a cubic foot of water by D, the load w_0 per inch of length at the top is

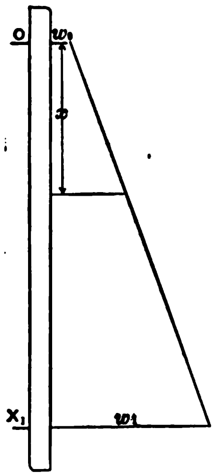

FIG. 217.

$$w_0 = \frac{Dh_0S}{12}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the load per inch of length at the bottom, where the depth is h_1, will be

$$w_1 = \frac{Dh_1S}{12}. \quad . \quad . \quad . \quad . \quad (1a)$$

The load at the distance x from the origin O will be

$$w = w_0 + (w_1 - w_0)\frac{x}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the total load on the frame is

$$W = \int_0^l w dx = w_0 l + \tfrac{1}{2}(w_1 - w_0)l = \tfrac{1}{2}(w_0 + w_1)l, \quad . \quad . \quad (3)$$

which might be inferred directly since the mean load per unit is

$$\tfrac{1}{2}(w_0 + w_1).$$

Let the supporting forces at the upper and lower ends be F_0 and F_1, which are unknown forces to be determined later. Then the shearing force at the distance x from the origin is

$$F = F_0 + \int_0^x \left[w_0 + (w_1 - w_0)\frac{x}{l} \right] dx = F_0 + w_0 x + (w_1 - w_0)\frac{x^2}{2l}. \quad (4)$$

If we represent the unknown bending moment at the origin by M_0, then the bending moment at the distance x from O is

$$M = M_0 + \int_0^x F dx = M_0 + \int_0^x \left(F_0 + w_0 x + \frac{w_1 - w_0}{2l} x^2\right) dx$$

$$= M_0 + F_0 x + \frac{w_0}{2} x^2 + \frac{w_1 - w_0}{6l} x^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By the theory of beams the second differential coefficient of the deflection v is

$$\frac{d^2v}{dx^2} = \frac{M}{EI},$$

where M is the bending moment, I is the moment of inertia of the section of the beam about its neutral axis, and E is the modulus of elasticity. Substituting for M from equation (5) and integrating,

$$\frac{dv}{dx} = \frac{1}{EI} \int_0^x \left(M_0 + F_0 x + \frac{w_0}{2} x^2 + \frac{w_1 - w_0}{6l} x^3\right) dx$$

$$= \frac{1}{EI}\left(M_0 x + F_0 \frac{x^2}{2} + \frac{w_0}{6} x^3 + \frac{w_1 - w_0}{24l} x^4\right). \quad . \quad . \quad . \quad . \quad (6)$$

A second integration gives for the deflection

$$v = \frac{1}{EI} \int_0^x \left(M_0 x + F_0 \frac{x^2}{2} + \frac{w_0}{6} x^3 + \frac{w_1 - w_0}{24l} x^4\right) dx,$$

$$v = \frac{1}{EI}\left(M_0 \frac{x^2}{2} + F_0 \frac{x^3}{6} + \frac{w_0}{24} x^4 + \frac{w_1 - w_0}{120l} x^5\right). \quad . \quad . \quad . \quad . \quad (7)$$

But since the frame is supposed to be fixed at the lower end, where $x = l$, we have at that point both the inclination $\frac{dx}{dv}$ and the deflection

v equal to zero, so that equations (6) and (7) give

$$0 = M_0 + F_0\frac{l}{2} + \frac{w_0}{6}l^2 + \frac{w_1 - w_0}{24}l^2, \quad . \quad . \quad . \quad . \quad .$$

and

$$0 = M_0 + F_0\frac{l}{3} + \frac{w_0}{12}l^2 + \frac{w_1 - w_0}{60}l^2. \quad . \quad . \quad . \quad . \quad .$$

Subtracting equation (9) from equation (8), we have

$$0 = \frac{1}{6}F_0 + \frac{w_0}{12}l + \frac{w_1 - w_0}{40}l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

from which

$$F_0 = -\frac{l}{20}(7w_0 + 3w_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

The negative sign shows that the supporting force is oppose the load, as it of course should be.

The supporting force at the lower end of the frame is obtai by subtracting the numerical value of F_0 from the total load W equation (3), so that

$$F_1 = -\frac{l}{20}(3w_0 + 7w_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Substituting for F_0 in equation (8) the value given by equa (11), we have

$$0 = M_0 - (7w_0 + 3w_1)\frac{l^2}{40} + \frac{w_0}{6}l^2 + \frac{w_1 - w_0}{24}l^2.$$

$$\therefore \quad M_0 = \frac{1}{20}w_0l^2 + \frac{1}{30}w_1l^2. \quad . \quad . \quad . \quad . \quad .$$

This gives us the means of calculating M_0 numerically, so th now becomes a known quantity, and we can then proceed to calcu the bending moment at any point by equation (5). For example, bending moment at the lower end of the frame will be

$$M_1 = M_0 + F_0l + \frac{w_0}{2}l^2 + \frac{w_1 - w_0}{6}l^2$$

$$= M_0 + F_0l + \left(\tfrac{1}{3}w_0 + \tfrac{1}{6}w_1\right)l^2. \quad . \quad . \quad . \quad .$$

This will always be the greatest bending moment on the frame. The bending moment decreases for sections above the lower end and becomes zero at the point of inflection. Near the middle of the beam the bending moment again attains a maximum, which can be determined by equating the first differential coefficient of the bending moment from equation (5) to zero; that is, by making F equal to zero in equation (4), so that

$$o = F_0 + w_0 x_m + (w_1 - w_0)\frac{x_m^2}{2l},$$

or

$$x_m^2 + \frac{2lw_0}{w_1 - w_0}x_m = -\frac{2l}{w_1 - w_0}F_0. \quad . \quad . \quad . \quad . \quad (15)$$

The numerical solution of this quadratic equation gives the value of x_m, which can be substituted in equation (5) to determine the corresponding bending moment.

To find the maximum deflection of the frame, we may equate to zero the first differential coefficient of the deflection from equation (7); that is, we may equate $\frac{dv}{dx}$ to zero from equation (6), giving

$$o = M_0 x_v + F_0\frac{x_v^2}{2} + \frac{w_0}{6}x_v^3 + \frac{w_1 - w_0}{24l}x_v^4,$$

from which

$$x_v^3 + \frac{4w_0 l}{w_1 - w_0}x_v^2 + \frac{12F_0 l}{w_1 - w_0}x_v = \frac{24M_0 l}{w_1 - w_0}. \quad . \quad . \quad . \quad (16)$$

The solution of this cubic equation gives the location of the maximum deflection at x_v, and the deflection can then be calculated for that point by aid of equation (7).

The only adequate way of securing the ends of a bulkhead frame is to use wide and deep brackets to secure the ends to the floor and to the deck. It now becomes a difficult question to determine what length of frame to use in calculating bending moment and deflection. One extreme will be to take only the frame between

the brackets; another extreme will be to consider that the brack merely make up for the lack of rigidity of floors and de and so take the entire distance between the floor and the de Probably something between these extremes will be fairest. any case the greatest stress will be found at that section of frame which is just above the lower bracket. It will be sho later that a close approximation to the deflection can be obtained the second extreme assumption namely, that the frame under c sideration extends from deck to floor, and that brackets only make up for lack of rigidity.

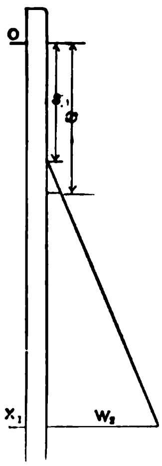

FIG. 218.

Having selected the section of the frame at wh the greatest stress is likely to be found, we may f the bending moment M for that section by equat (5), and then the stress can be found in the usual w by the equation

$$\sigma=\frac{My}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

It may happen that a compartment will not be e tirely filled, or it may be desirable to determi beforehand the extent to which a compartment c be safely filled when a bulkhead is tested under wa pressure; for it is unwise to distort or give a perm nent set to a bulkhead during such a test. Fig. 218 represents a bu head frame with the water a distance s from the upper end.

The load per inch of length at a distance x (greater than s) fr the origin will now be

$$w=w_2\frac{x-s}{l-s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

and the total load on the frame is

$$w=\tfrac{1}{2}w_2(l-s). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

The shearing force will be F_0 at the upper end of a bulkhead frame, as far as to the surface of the water. Below the surface of the

water the shearing force will be

$$F = F_0 + \int_0^{x-s} w_2 \frac{x-s}{l-s} d(x-s) = F_0 + \frac{w_2(x-s)^2}{2(l-s)} . \quad . \quad . \quad (20)$$

The bending moment above the water surface will be found by adding F_0x to the bending moment M_0 over the support. Below the surface of the water the bending moment will be

$$M = M_0 + \int_0^x F_0 dx + \int_0^{x-s} \frac{w_2}{2(l-s)} (x-s)^2 d(x-s)$$

$$= M_0 + F_0 x + \frac{w_2(x-s)^3}{6(l-s)} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The slope of the beam is

$$\frac{dv}{dx} = \frac{1}{EI} \left[\int_0^x (M_0 + F_0 x) dx + \frac{w_2}{6(l-s)} \int_0^{x-s} (x-s)^3 d(x-s) \right]$$

$$= \frac{1}{EI} \left(M_0 x + \tfrac{1}{2} F_0 x^2 + \frac{w_2 (x-s)^4}{24(l-s)} \right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The deflection of the frame is

$$v = \frac{1}{EI} \left[\int_0^x (M_0 x + \tfrac{1}{2} F_0 x^2) dx + \frac{w_2}{24(l-s)} \int_0^{x-s} (x-s)^4 d(x-s) \right],$$

$$v = \frac{1}{EI} \left(\tfrac{1}{2} M_0 x^2 + \tfrac{1}{6} F_0 x^3 + \frac{w_2 (x-s)^5}{120(l-s)} \right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

But at the lower end of the frame, where $x = l$, both slope and deflection are zero, so that

$$0 = M_0 l + \tfrac{1}{2} F_0 l^2 + \frac{w_2 (l-s)^3}{24},$$

and

$$0 = M_0 l + \tfrac{1}{3} F_0 l^2 + \frac{w_2 (l-s)^4}{60 l},$$

and subtracting gives

$$o=\tfrac{1}{6}F_0l^3+w_2\frac{(3l+2s)\,(l-s)^3}{120l},$$

$$F_0=-\tfrac{1}{20}w_2\frac{(3l+2s)\,(l-s)^3}{l^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

It will be convenient to calculate F_0 numerically, and insert it in the equations where it appears.

$$V=\frac{1}{EI}\left[\int_0^x (M_0x-\tfrac{1}{2}F_0x^2)dx+\frac{w_2}{24(l-s)}\int_0^{x-s}(x-s)^4d(x-s)\right],$$

$$V=\frac{1}{EI}\left(\tfrac{1}{2}M_0x^2-\tfrac{1}{6}F_0x^3+\frac{w_2(x-s)^5}{120(l-s)}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The maximum deflection is obtained by using for x in the above equation the value obtained by equating $\frac{dv}{dx}$ to zero in equation (22), which gives

$$o=M_0x_v-\tfrac{1}{2}F_0x_v^2+\frac{w_2(x_v-s)^4}{24(l-s)};$$

consequently

$$x_v^4-4x_v^3s+\left[6s^2-\frac{12}{w_2}F_0(l-s)\right]x_v^2+\left[24\frac{(l-s)}{w_2}M_0-4s^3\right]x_v+s^4=o. \quad (26)$$

To illustrate the use of this method of calculating the strength of a bulkhead it will be applied to the longitudinal bulkhead separating the engine-rooms of the U. S. battle-ship *Illinois*, which was tested under the direction of Naval Constructor J. J. Woodward, U.S.N.*

The frames each consisted of two members, one on each side of the plating. Each member consisted of a Z bar 6 inches deep, and with flanges 3 inches wide, weighing 15 pounds per foot of length, reinforced by an angle-bar with a web 4 inches deep and a flange 3 inches wide, and weighing 9 pounds per foot of length. Allowing for the thickness of the plating between the two members, the frames had a total depth of 12¾ inches, and had flanges 6 inches wide. The total moment of inertia about the neutral axis was 243.

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The frames were secured by brackets to the floors of the double bottom at the lower end, and to the beams under the armored deck at the upper ends by triangular brackets of 15-pound plate. The brackets on the two sides of a frame were dissimilar at both the top and the bottom, depending on the placing of piping, machinery, etc. The brackets at the bottom were 16½ inches wide; one was 4 feet high and the other 15 inches high. At the top one bracket was 20 inches deep and 15 inches wide; the other was about 24 inches deep and 17 inches wide, but the channel-bar was cut short on this side to clear the work for securing a water-tight joint between the bulkhead and the armored deck. The reinforcing angle-irons were carried along the outside edges of all the brackets to stiffen them, and the deep bracket at the lower end was reinforced by an angle-iron clip 3 inches by 3 inches and weighing 7 pounds per foot.

The total length of the frames from the lower edge of the deck-beams to the floors was 22 feet 8 inches (272 inches), and the head of water for the test was 24 feet above the floors. The following computation is made on the assumption that the brackets at the top and the bottom of a bulkhead frame compensates for lack of rigidity of the floor and deck, taking the entire length of the frame for l.

The frames were spaced 4 feet apart; consequently the load on a frame per inch of length at the lower end was

$$w_1 = 62.4 \times 24 \times 4 \div 12 = 499.2 \text{ pounds,}$$

where 62.4 is the weight of a cubic foot of fresh water.

The head of water acting at the top of a frame was

$$24 - 22\tfrac{8}{12} = 1.33 \text{ feet,}$$

and the load per inch of length was

$$w_0 = 62.4 \times 1.33 \times 4 \div 12 = 27.7 \text{ pounds.}$$

The total load on the beam, by equation (3), was

$$w = (27.7 + 499.2)\tfrac{272}{2} = 71660 \text{ pounds.}$$

From equations (11) and (12) the supporting forces at the upper and the lower ends are

$$F_0 = -\tfrac{272}{20}(7\times 27.7 + 3\times 499.2) = -23000 \text{ pounds,}$$
$$F_1 = -\tfrac{272}{20}(3\times 27.7 + 7\times 499.2) = -48660 \text{ pounds.}$$

The bending moment at the top of the frame is, by equation (13),

$$M_0 = \frac{27.7\times 272^2}{20} + \frac{499.2\times 272^2}{30} = +1333000.$$

The section which has the maximum bending moment near the middle of the frame is found by equation (15):

$$x_m + 2\frac{272\times 27.7}{471.5}x_m = \frac{2\times 23000\times 272}{471.5}; \quad \therefore x_m = 147.7 \text{ inches.}$$

At this section the bending moment is, by equation (5),

$$M = 1333000 - 23000\times 147.7 + \frac{27.7}{2}147.7^2 + \frac{471.5}{6\times 272}\times 147.7^3$$
$$= -830000 \text{ inch-pounds.}$$

The stress on the most strained fibre at this section is

$$\sigma = \frac{My}{I} = \frac{830000\times 6\tfrac{3}{8}}{292} = 18000 \text{ pounds per square inch;}$$

where 292 is the moment of inertia of the section and $6\tfrac{3}{8}$ is the distance of the most strained fibre.

The bending moment at the bottom of a frame is given by equation (14):

$$M_1 = 1333000 - 23000\times 272 + \left(\frac{27.7}{3} + \frac{499.2}{6}\right)272^2 = +1910000.$$

This is much greater than the maximum bending moment near the middle of the frame, but the moment of inertia is so great, due to the two large brackets, that the fibre stress will be relatively low.

The section just at the top of the large lower bracket is

$$272 - 4\times 12 = 224$$

inches from the top of the frame; the bending moment at this section is consequently, by equation (5),

$$M = 1333000 - 23000 \times 224 + \frac{27.7}{2} \times 224^2 + \frac{471.5}{6 \times 272} 224^3$$
$$= -5152000 + 5276000 = +124000.$$

The fibre stress occasioned by this bending moment will be small in comparison with that near the middle of the frame, since the moment of inertia is the same for both sections.

It would probably be advisable to calculate the fibre stress at some section between the top and bottom of the lower bracket, although it is doubtful if it would be found greater than that near the middle of the frame.

The distance from the top to the section of maximum deflection is given by equation (16):

$$x_v^3 + \frac{4 \times 27.7 \times 272}{471.5} x_v^2 - \frac{12 \times 23000 \times 272}{471.5} x_v + \frac{24 \times 1333600 \times 272}{471.5} = 0,$$
$$x_v = 142 \text{ inches.}$$

At this section the deflection is, by equation (7),

$$v = \frac{142^2}{28000000 \times 292} \left(\frac{1333000}{2} - \frac{23000 \times 142}{6} + \frac{27.7 \times 142^2}{24} + \frac{471.5}{120 \times 272} 142^3 \right),$$
$$v = 0.499 \text{ inches,}$$

provided the value of E, the modulus of elasticity, is taken as 28000000. The greatest deflection as actually measured on the frame was at a depth of 129 inches, and was equal to $\frac{13}{16}$ of an inch. The discrepancy is due in part to the lack of rigidity of the riveted construction of the bulkhead frame.

Bulkhead Plating.—In discussing the strength of bulkhead plating two cases can be distinguished. If the bulkhead is subjected to pressure first on one side and then on the other, as may occur in an oil-carrying steamer, it appears wise to make the plating stiff enough to avoid a permanent set. But if a bulkhead (such as one aft of the engine-room) will be subjected to pressure only through

an accident, it will be sufficient to make sure that the plating will not be ruptured or so distorted as to leak badly. Bulkhead plating which is always subjected to pressure on one side, as for trimming tanks, may be treated by the second method, provided that the plating is thick enough to avoid an increasing amount of permanent set; but it would appear to be better to treat such plating by the first method.

For the present purpose it will be assumed that the framing of the bulkhead is strong enough to carry the load due to water pressure when a compartment on one side of it is flooded, and that, in general, the framing will consist of vertical members fixed at the ends. The plating between such vertical members, much like flooring or floor girders in a building, is expected only to carry the load to the framing. It will be sufficient to consider a horizontal strip one inch wide reaching from one frame to the next. This strip of plating may be considered to be a beam fixed at the ends and uniformly loaded. If W is the entire load on the beam, then by the ordinary theory of beams the bending moment is greatest at the supports and is equal to

$$M = \tfrac{1}{12} Wl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where l is the length of the strip in inches. The moment of resistance is

$$\frac{\sigma I}{y},$$

as in the preceding investigation, σ being the stress, I the moment of inertia of the section of the beam, and y is the most strained fibre. Taking unity for the width, and t for the thickness of the strip of plating, then

$$I = \tfrac{1}{12} t^3 \quad \text{and} \quad y = \tfrac{1}{2} t,$$

so that

$$\tfrac{1}{12} Wl = \tfrac{1}{6} \sigma t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\sigma = \frac{1}{2} \frac{Wl}{t^2}, \quad \text{or} \quad t = \sqrt{\frac{1}{2} \frac{Wl}{\sigma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The deflection of the plating under the load W is

$$v=\frac{\sigma l^2}{32Ey}=\frac{\sigma l^2}{16Et}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where E is the modulus of elasticity, and y is the half-thickness of the plate.

So long as the stress at the supports, as calculated by equation (3), does not exceed the elastic limit of the material of which the plating is made, then when the pressure is released the plate will again become flat as it was before the pressure was applied.

If the elastic limit is exceeded at the supports, the metal will begin to flow at that place, that is, the plate will begin to bend around the edge of the support. The effect of this bending is to increase the deflection so that the plate begins to bulge under the pressure of the water on it. If this is continued far enough, the plate will bulge into a cylindrical form between the frames, and will then be subjected to tension only, as is the plate of a cylindrical steam-boiler. It may readily happen, however, that in the process of bulging the plate the elastic limit has been exceeded only at the edge of the frame, and that consequently the plate will tend to flatten out when the water pressure is removed. But the plate cannot become flat at the edge of the frame, for the sharp bend or kink at that place will remain after the pressure is entirely removed. It does not appear that the plating will be injured by repeated applications of pressure on the same side; but if pressure is applied first on one side and then on the other, the plate will be bent back and forth at the edge of the frame, and will finally become hard and brittle, so that it will be likely to crack and fail at that point, more especially if corrosion is set up in the crack. Consequently it may be unwise to use this method for dealing with plating on a bulkhead which separates the compartments of an oil-carrying steamer. And yet, as the angle to which the plate is bent is small, it will probably endure bending back and forth very many times before a crack is started.

A thin plate which has bulged into a cylindrical form without exceeding the elastic limit (save at the supports) may be likened to an elastic, flexible cord which is just long enough to reach between

two supports when it is not extended by its weight. Such a cord will of course hang in a catenary under the influence of gravity and its elasticity. The thin plate of course takes a cylindrical form because the pressure on it is uniform; the radius of the cylinder will depend on the pressure, the thickness, and the elasticity of the plate and the distance between the frames.

In order to find the relation between the hydrostatic pressure on the plate and the thickness to withstand that pressure we may proceed in the following way: If σ is the tension per square inch of the plate, and E is the modulus of elasticity, then the stretch per unit of length is $\sigma \div E$. If the distance from an edge of a frame to the nearest edge of the next frame is s, then the stretch of the plate between two frames is $s\sigma \div E$. Before the plate is bulged under pressure it will lie flat, and its length s will form a chord reaching from frame to frame; after it is bulged it will form an arc having the length

$$s + s\frac{\sigma}{E} = s\left(1 + \frac{\sigma}{E}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5$$

The corresponding radius cannot be determined directly from the usual trigonometric functions, but may be obtained by interpolation in the following table, which has been calculated for the purpose.

PROPERTIES OF CIRCULAR ARCS. CHORD = UNITY.

Angle.	Length of Arc.	Length of Radius.	Rise of Arc.	Stress Pounds per Square Inch.
1.	2.	3.	4	5.
5°	1.000317	11.463	0.0110	8900
5° 30′	1.000384	10.421	0.0120	10800
6°	1.000457	9.554	0.0131	12800
6° 30′	1.000537	8.819	0.0142	15000
7°	1.000622	8.190	0.0153	17400
7° 30′	1.000715	7.645	0.0164	20000
8°	1.000813	7.168	0.0175	22800
8° 30′	1.000918	6.747	0.0186	25700
9°	1.001029	6.373	0.0196	28800
9° 30′	1.001145	6.038	0.0207	32100

Thus an arc which subtends an angle of 5° is 1.000317 times as long

as the chord, and its radius is 11.463 times the length of the chord. The stretch is

$$0.000317s = s\frac{\sigma}{E},$$

so that the tension is

$$\sigma = 0.000317 \times E = 0.000317 \times 28000000 = 8900$$

pounds per square inch for medium mild steel.

By aid of the table the radius corresponding to a given working tension σ can be readily determined, and then the thickness can be found by the usual equation for a thin hollow cylinder, which gives

$$t = \frac{pr}{\sigma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where r is the radius in inches, p is the fluid pressure in pounds per square inch, and σ is the safe tensional strength.

The deflection of the plate can be found by multiplying the rise of the arc by the distance between frames.

The frames, or stiffeners, for important bulkheads should always be made with a member (a Z bar or other rolled form) on each side of the plating, so that the rivets may come near the neutral axis of the section of the frame. The plate between the two members of a frame will take part in the stretching when the bulkhead is subjected to pressure, the amount and location of the stretch depending on the riveting of the frame to the plating. The effect of the stretching of this part of the plating is difficult to determine, but as it tends to decrease the radius, and consequently the stress in the plate after it has assumed a cylindrical form, we can well afford to neglect it.

Suppose that it is desired to limit the stress in the plating to 15000 pounds per square inch. Then, from the table of the Properties of Circular Arcs, it appears that the radius of the plate after it is bulged into a circular form near the bottom of a frame is 8819 times the chord. This gives for the radius

$$45 \times 8.819 = 39.7 \text{ inches,}$$

if the distance between the frames is 45 inches from edge to edge

as in the bulkhead described on page 601, because then the spacing from centretto centre was four feet and the flange of the Z bars was 3 inches wide. Now the head of water during the test was 24 feet above the floors, giving a pressure of

$$p=24\times62.4\div144=10.4 \text{ pounds per square inch;}$$

consequently the thickness of the plate by equation (6) should be

$$t=10.4\times39.7\div15000=0.028 \text{ of an inch.}$$

If it is desired that the stress shall not exceed 15000 pounds by the ordinary beam theory, equation (3) gives for the thickness

$$t=\{\tfrac{1}{2}\times10.4\times45^2\div15000\}^{\frac{1}{2}}=0.83 \text{ of an inch.}$$

Now the plating used for the bulkhead weighed 15 pounds per square foot, so that if a plate an inch thick is assumed to weigh 40 pounds to the square foot, the plating was three-eighths of an inch thick.

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